Qualitative behavior of SIS epidemic model on time scales

Wichuta Sae-jie, Kornkanok Bunwong, and Elvin J. Moore

Abstract**—**Mathematical models in continuous or discrete time are widely used to simplify real-world systems in order to understand their mechanisms for a particular purpose. Consequently, a well-defined model should be able to carry out some predictions and be fitted to observational data in a variety of time measurements (seconds, hours, days, weeks, months, or years). Therefore, the time scales approach also plays an important role in the model. In this paper, we construct a time scales version of a simple epidemic model (SIS) and explore the variety of its qualitative behavior. For each parameter value, the theory of time scales allows the discovery of similar and dissimilar behavior of SIS epidemic models on different time scales. Finally, the dynamic behavior shows a period doubling bifurcation path to chaos as the distance of equally spaced points in time increases.

Keywords**—** Bifurcation, Chaos, Limit cycles, Period doubling, SIS epidemic model, Time scales analysis.

I. Introduction

THE theory of time scales has recently received a lot of attention. First, Stefan Hilger introduced this theory in order to unify continuous ($\mathbb{R}$) and discrete ($\mathbb{Z}$) analysis [1]. Since then, the theory has been extended. Nowadays time scales theory can be used to explain not only continuous and discrete times but also other types of time. Time scales theory has been used in the study of first order dynamic equations [2], first order dynamical systems [3]-[5], numerical results [6], and a variety of mathematical models, including a plant population model [7], economic model [8], predator-prey model [9], and West Nile virus model [10]. In this paper, the time scales theory is used to analyze the qualitative behavior of an SIS epidemic model as the time scale is changed.

An SIS epidemic model is such a very well known disease transmission model for diseases in which the disease does not produce immunity. The population in this model is divided into a susceptible (*S*) group and an infectious (*I*) group, with the *S* group becoming infected by the *I* group and the *I* group recovering from the disease and returning to the *S* group. Some diseases which follow this pattern are: some STD's (e.g., chancroid), the eye disease haemorrhagic conjunctivitis, the common cold. These SIS epidemic models have been studied for continuous ($\mathbb{R}$) time by using differential equation models [11]. For a continuous time model in which the total population size is assumed to be constant, the solution trajectory tends either to an endemic or a disease free equilibrium point depending on the basic reproduction number. In addition, an SIS model with constant population size can be reduced to a one-variable model in which the behavior of a solution is always non-oscillatory and therefore oscillation cannot persist for the endemic disease region.

For discrete ($\mathbb{Z}$) time, the SIS model is constructed by using difference equation models, as in [12]-[17]. In this case, the behavior of the endemic solution can be very complicated as it can tend to an equilibrium point, to limit cycles or show chaos.

An outline of this paper is as follows. In section II, we review the theory of time scales. In section III, we develop an SIS epidemic model using the theory of time scales. In section IV, we give a qualitative analysis of the SIS time-scale model. In section V, we show numerical results for the SIS model. In section VI, we discuss the results and draw conclusions.

II. Basic definitions on time scales

A time scale is an arbitrary nonempty closed subset of the real numbers [7]. A time scale is usually denoted by the symbol $\mathbb{T}$. Forward and backward jump operators are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf\varnothing = \sup\mathbb{T}$, $\sup\varnothing = \inf\mathbb{T}$ and $\varnothing$ denotes the empty set. A point $t \in \mathbb{T}$ is called *left-dense* if $t > \inf\mathbb{T}$ and $\rho(t) = t$, *right-dense* if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$ and *right-scattered* if $\sigma(t) > t$ as summarized in Table I.

The set $\mathbb{T}^{\kappa}$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum m; otherwise it is $\mathbb{T}$ without this left-scattered maximum. The *graininess function* $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$. Moreover, a function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* provided f is continuous at right-dense points and left-hand limits exist and it is finite at left-dense points in $\mathbb{T}$.

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TABLE I
CLASSIFICATION OF POINTS

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

The (*delta*) derivative of $f : \mathbb{T} \to \mathbb{R}$ at point $t \in \mathbb{T}^\kappa$ is defined as follows. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then $f^\Delta(t)$ is defined to be the number (provided it exists) with the property that for all $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \le \varepsilon \left| \sigma(t) - s \right|,$$

for all $s \in U$.

Another useful formula for the relationship concerning the (delta) derivative is given by

$$f^\Delta(t) = \begin{cases} \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s} & if\ \mu(t) = 0 \\ \dfrac{f(\sigma(t)) - f(t)}{\sigma(t) - t} & if\ \mu(t) > 0. \end{cases} \quad (1)$$

To avoid separate discussion of the two cases $\mu(t) = 0$ and $\mu(t) > 0$, there is another useful formula, which holds when f is delta differentiable at $t \in \mathbb{T}^\kappa$:

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t). \quad (2)$$

In this paper we mainly focus on three different time scales as visualized in Fig. 1.

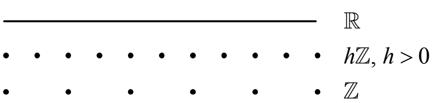

Fig.1 An example of time scales.

In the case $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = \rho(t) = t\ ,\ \mu(t) = 0\ ,\ f^\Delta(t) = f'(t)\ ,\ \int_m^n f(t)\Delta t = \int_m^n f(t)dt\ ,$$

where $f'(t) = df(t)/dt$ is the right-derivative of $f(t)$. Thus, the time scales operators reduce to the corresponding continuous operators.

In the case $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}, h > 0\}$, i.e., where the points are equally spaced points in time, we have

$$\sigma(t) = t + h\ ,\ \rho(t) = t - h\ ,\ \mu(t) = h\ ,$$

$$f^\Delta(t) = \frac{f(t+h) - f(t)}{h}\ ,\ \int_m^n f(t)\Delta t = \sum_{t=m}^{n-1} h f(t)\ .$$

Obviously, $\mathbb{T} = \mathbb{Z}$ is a particular case when $h = 1$.

Therefore, every point is isolated. This time scales is considered as discrete and the delta derivative operator corresponds to the forward finite difference operator (Δ) and the delta integration corresponds to summation.

Assume $t_0 \in \mathbb{T}$ and it is convenient to let $t_0 > 0$. The time scale interval $[0, t_0)_\mathbb{T}$ is defined by $[0, t_0)_\mathbb{T} = [0, t_0)_\mathbb{T} \cap \mathbb{T}$. The nontrivial function, $z(t)$, is called the solution of the dynamic system

$$z^\Delta(t) = f(t, z(t)), \quad z \in \mathbb{R}^n, \quad t \in \mathbb{T} \quad (3)$$

when $z(t) \in C^1_{rd}([t_0, \infty)_\mathbb{T}, \mathbb{R}^n)$ and satisfies (3). If $z(t)$ also satisfies the initial condition

$$z(t_0) = z_0, \quad (4)$$

then $z(t)$ is called the solution of initial value problem (3) and (4).

III. SIS EPIDEMIC MODELS ON TIME SCALES

For the SIS epidemic model on time scales $\mathbb{T}$, $S(t)$ represents the number of susceptible individuals at time t and $I(t)$ is the number of infectious individuals at time t. We assume that the total population size is a constant N. Let γ be the recovery rate of an infectious individual who then returns to the susceptible population. Then $\gamma I(t)$ represents the total number of infectious individuals who recover per unit time at the time t. Let α be the disease virulence per unit time, i.e., the rate of infection of a susceptible person due to contact with an infectious person. Then $(\alpha / N) S(t) I(t)$ represents the infection rate at which the susceptible population contracts the disease at time t. Thus the SIS epidemic system for time scales can be written in the following form:

$$S^\Delta(t) = I(t)\left(-\frac{\alpha}{N} S(t) + \gamma\right),\ S(t) \ge 0 \quad (5)$$

$$I^\Delta(t) = I(t)\left(\frac{\alpha}{N} S(t) - \gamma\right),\ I(t) \ge 0 \quad (6)$$

with positive initial conditions $S(0)$ and $I(0)$ satisfying $S(0) + I(0) = N$. The total population size remains constant and thus

$$S(t) + I(t) = S(\sigma(t)) + I(\sigma(t)) = N \ \text{ for } \ t \ge 0.$$

Another assumption is that the population is homogeneously mixed at all times. The parameters α, γ, N are all positive constants.

For the continuous time scale, the system becomes

$$\frac{dS}{dt} = I(t)\left(-\frac{\alpha}{N} S(t) + \gamma\right), S(t) \ge 0 \quad (7)$$

$$\frac{dI}{dt} = I(t)\left(\frac{\alpha}{N} S(t) - \gamma\right),\ I(t) \ge 0 \quad (8)$$

The system can be changed to be a single equation by substituting $I(t) = N - S(t)$ into (7). Therefore,

$$\frac{dS}{dt} = \frac{\alpha}{N} S^2(t) - (\alpha + \gamma) S(t) + \gamma N \quad (9)$$

An exact solution of (9) can be obtained by integrating ds/dt using the method of partial fractions. The result is:

$$S(t) = \left[\frac{C_0 \gamma}{\alpha} e^{(\alpha - \gamma)t} - 1\right] N \Big/ \left[C_0 e^{(\alpha - \gamma)t} - 1\right] N,$$

where $C_0 = (S_0 - N)/(S_0 - \dfrac{\gamma N}{\alpha})$ and $S_0 = S(0)$.

Obviously, the asymptotic behavior of $S(t)$ for large t is

$$S(t)=\begin{cases}\dfrac{\gamma N}{\alpha} & \text{for } \alpha>\gamma\\ N & \text{for } \alpha<\gamma\end{cases}$$

Therefore, the solution of (9) is non-oscillatory and reaches an equilibrium point.

In the case $\mathbb{T}=h\mathbb{Z}=\{hk: k\in\mathbb{Z}, h>0\}$, the graininess function is defined by $\mu(t)=(t+h)-t=h$. Therefore, the SIS epidemic model is

$$S^{\Delta}(t)=\frac{S(\sigma(t))-S(t)}{\mu(t)}=\frac{S(t+h)-S(t)}{h}=I(t)\left(-\frac{\alpha}{N}S(t)+\gamma\right),\ S(t)\ge 0. \quad (10)$$

$$I^{\Delta}(t)=\frac{I(\sigma(t))-I(t)}{\mu(t)}=\frac{I(t+h)-I(t)}{h}=I(t)\left(\frac{\alpha}{N}S(t)-\gamma\right),\ I(t)\ge 0. \quad (11)$$

The system can be changed to a single equation as before

$$S^{\Delta}(t)=\frac{\alpha}{N}S^2(t)-(\alpha+\gamma)S(t)+\gamma N \quad (12)$$

For the discrete time scale, the equation can be written as a difference equation:

$$S(\sigma(t))=\frac{\alpha\mu}{N}S^2(t)+(1-\alpha\mu-\gamma\mu)S(t)+\gamma N\mu := f(S(t)). \quad (13)$$

The analytical solution of (13) for all values of parameters is still unknown, although numerical solutions can be obtained for any given parameter values. Therefore, qualitative analysis is a useful tool.

IV. Qualitative Analysis of SIS Epidemic Models

A. Equilibrium Points

For a natural disease process, each parameter is assumed to be positive and each variable is non-negative. Therefore, the region of interest is

$$\Gamma=\{(S,I)\in\mathbb{R}^2 \mid S\ge 0, I\ge 0, S+I=N\}$$

The equilibrium point or the steady state (time-independent) solution is obtained by setting $S(\sigma(t))=S(t)=f(S(t))=S^*$ in (13).

$$\frac{\alpha}{N}S^2-(\alpha+\gamma)S+\gamma N=0. \quad (14)$$

Therefore, $S^*_{1,2}=\dfrac{\left((\alpha+\gamma)\pm|\alpha-\gamma|\right)N}{2\alpha}$.

There are two equilibrium points for both $\alpha<\gamma$ and $\alpha>\gamma$, namely, the disease-free equilibrium point $(S_1^*, I_1^*)=(N,0)$ and the endemic equilibrium point $(S_2^*, I_2^*)=(\gamma N/\alpha, N-\gamma N/\alpha)$. However, this second equilibrium point only satisfies the conditions $0<S(t)<N$, $0<I(t)<N$ when $\alpha>\gamma$. Consequently, we first consider $\alpha>\gamma$.

B. Stability

We consider a first-order dynamic equation in the following form:

$$x^{\Delta}(t)=F(t,x(t))$$

where $x(t)$ is the value of x at time t.

The conditions for asymptotic stability of equilibrium points, x^*, are obtained by linearization of the equations [18]. For the discrete time scale, the condition is that $|dF/dx|<1$, where $x=x^*$, and for the continuous time scale, the condition is that the real part of $\dfrac{dF}{dx}<0$, where $x=x^*$.

To determine the asymptotic stability of the discrete case (13) we look at $S(t)$ close to S^* where S^* is the equilibrium point and define

$$S(t)=S^*+\tilde{S}(t) \quad (15)$$

where $\tilde{S}(t)$ is a small quantity termed a perturbation of the equilibrium point S^*. Then,

$$\tilde{S}(\sigma(t))=S(\sigma(t))-S^*=f(S(t))-S^*=f(S^*+\tilde{S}(t))-S^* \quad (16)$$

and a Taylor series expansion of $f(S(t))$ about the point S^* gives:

$$f(S^*+\tilde{S}(t))=f(S^*)+\left(\left.\frac{df}{dS}\right|_{S^*}\right)\tilde{S}(t)+O(\tilde{S}^2(t)).$$

$O(\tilde{S}^2(t))$ is very small and can be neglected. The linear approximation for (16) is:

$$\tilde{S}(\sigma(t))=\left(\left.\frac{df}{dS}\right|_{S^*}\right)\tilde{S}(t)=a\tilde{S}(t).$$

Thus, if $|a|<1$, then the equilibrium point is asymptotically stable.

Theorem 1. If the inequalities $-2<(\alpha-\gamma)\mu<0$ hold, then a disease-free equilibrium point $S_1^*=N$ is locally asymptotically stable. Otherwise, $S_1^*=N$ is unstable.

Proof. For (13), the asymptotic stability is given by

$$|a|_{S_1^*=N}=\left|\frac{2\alpha\mu S}{N}+(1-\alpha\mu-\gamma\mu)\right|_{S_1^*=N}=|1+\mu(\alpha-\gamma)|<1.$$

Theorem 2. If the inequality $0<(\alpha-\gamma)\mu<2$ holds, then an endemic equilibrium point $S_2^*=\gamma N/\alpha$ is locally asymptotically stable. Otherwise, $S_2^*=\gamma N/\alpha$ is unstable.

Proof. For (13), the stability is determined by

$$|a|_{S_2^*}=\left|\frac{2\mu\alpha}{N}\frac{\gamma N}{\alpha}+(1-\alpha\mu-\gamma\mu)\right|=|1+\mu(\gamma-\alpha)|<1.$$

C. Initial Conditions

Lemma 1. For $(\alpha+\gamma)\mu-1<0$, $\gamma\mu<1$ and $\alpha>\gamma$, the solutions to the single-population SIS model are positive for all initial conditions $[0,N]$.

Lemma 2. The solutions to SIS epidemic model are positive for all initial conditions if and only if $0\le 1-\gamma\mu<\alpha\mu<(1+\sqrt{\gamma\mu})^2$ and $\alpha>\gamma$.

Proof. It is similar to the proof in [12].

Lemma 3. For $(\alpha+\gamma)\mu-1>0$, $\alpha>\gamma$, and $\gamma\mu>1$, the solutions to the single population SIS model are positive for initial conditions $\left(\dfrac{N(\gamma\mu-1)}{\alpha\mu}, N\right)$.

Proposition 1. The solutions of the SIS epidemic model of (5) and (6) remain nonnegative and are bounded under conditions stated in Theorems 1, 2 and Lemmas 1-3.

D. The Period-Doubling Route to Chaos

To find the period two cycle, we need to find the solutions of $f(f(S(t))=S(t)$. In addition to the equilibrium point of (13) given by $S(t)=S(\sigma(t))=f(S(t))$, there are two more equilibrium points of (13) given by

$$S(t)=f(S(\sigma(t)))=f(f(S(t)))=f^2(S(t)),$$

which form a period 2 cycle. Therefore, the two points on the period 2 cycle are

$$\overline{\overline{S}}_{1,2}=\left(\alpha\mu+\gamma\mu-2\pm\sqrt{(\alpha-\gamma)^2\mu^2-4}\right)N\Big/(2\alpha\mu).$$

The period two cycle exists when the square root is real, i.e., when $(\alpha-\gamma)\mu>2$.

The stability is determined by $\left|a|_{S=\overline{\overline{S}}_1}\right|\left|a|_{S=\overline{\overline{S}}_2}\right|<1$.

From this condition it can be shown that the period 2 cycle $\overline{\overline{S}}_{1,2}$ is locally asymptotically stable if $2<(\alpha-\gamma)\mu<\sqrt{6}$, Otherwise the cycle $\overline{\overline{S}}_{1,2}$ is unstable.

To find the period 2^n cycle, let $f^{2^n}(S(t))=S(t)$ and solve for equilibrium points ($\hat{S}$). The stability is considered by $\left|a|_{S=\hat{S}_1}\right|\left|a|_{S=\hat{S}_2}\right|\ldots\left|a|_{S=\hat{S}_n}\right|<1$. It is extremely complicated, if not impossible, to find these higher-order limit cycles by analytical methods, and therefore numerical methods are useful.

V. NUMERICAL RESULTS

The SIS epidemic model exhibits various dynamical behaviors in respect of the number of susceptible individuals, if the bifurcation parameter μ exceed certain values as shown in Fig. 2. The number of susceptible individuals shows many different equilibrium states (infinite number of possibilities), for certain high values of the parameter as shown in a hierarchy of bifurcations.

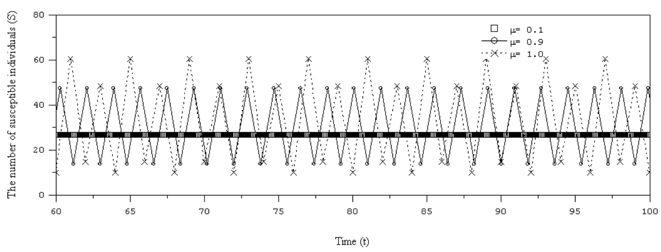

Fig. 2 The time series solution of (13) with $\alpha=3.4$, $\gamma=0.9$. The solutions appear as asymptotically stable, a period two cycle, a period four cycle when $\mu=0.1, 0.9, 1.0$ respectively.

Fig. 3 shows that the equilibrium is a stable point for all $\mu\in[0,1.13]$, $\alpha=2$, $\gamma=0.9$, and $N=100$. For these parameter values, the continuous SIS model and the discrete SIS model give the same behavior for the solution, namely one asymptotically stable equilibrium point. The solution tends to the equilibrium point $S_2^*=45$.

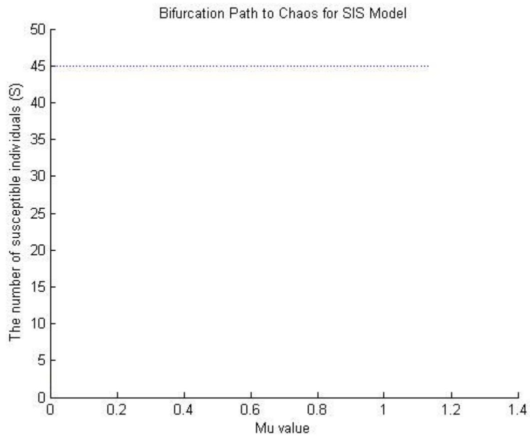

Fig. 3 The bifurcation diagram of μ. The parameter values in (13) are: $\alpha=2$, $\gamma=0.9$. There is one asymptotically stable equilibrium point.

Fig. 4 shows the solution behavior of (13) for $\alpha=3.2$, $\gamma=0.9$. For $\mu\in(0,0.869565)$, the solution is non-oscillatory and the behavior is the same as the continuous SIS model. At $\mu=1$, the SIS model exhibits periodic behavior with a period two cycle. At higher values of μ the solutions show a bifurcation path to chaos.

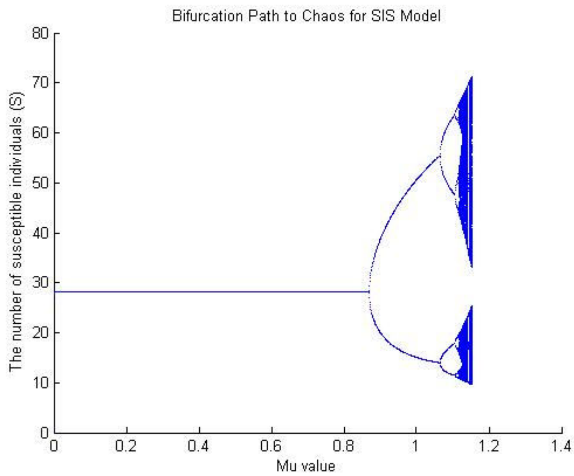

Fig. 4 The bifurcation diagram of μ. The parameter values in (13) are : $\alpha=3.2$, $\gamma=0.9$, and $N=100$.

Fig. 5 shows the solution behavior of (13) when $\alpha=3.4$, $\gamma=0.9$. For $\mu\in(0,0.8)$ the solution is non-oscillatory. For these parameter values, the SIS model exhibits periodic behavior with a period four cycle at $\mu=1$.

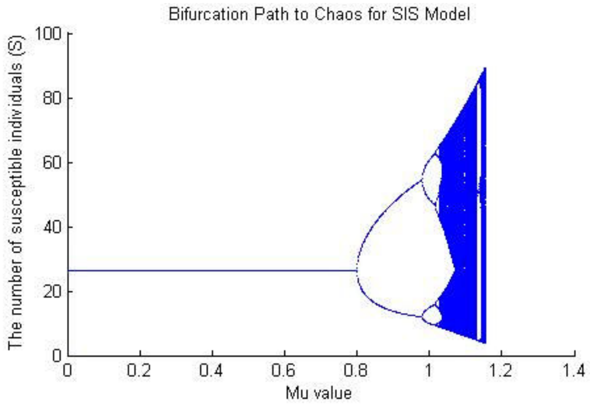

Fig. 5 The bifurcation diagram of μ. The parameter values in (13) are: $\alpha = 3.4$, $\gamma = 0.9$, and $N = 100$.

The period four cycle can be obtained by numerical computation of the equilibrium points of $f^4(S(t)) = S(t)$. For parameter values $\alpha = 3.6$, $\gamma = 0.9$, and $N = 100$, The system has a stable period two cycle when $\mu \in (0.740741, 0.907218)$, and a stable period four cycle when $\mu \in (0.907218, 0.942256)$.

From [14], [19], the SIS epidemic model can be transformed to the discrete logistic model

$$x(\sigma(t)) = f(x(t)) = rx(t)\left(1 - x(t)\right),$$

by the substitutions $x(t) = \dfrac{\alpha\mu I(t)}{N(1 - \gamma\mu + \alpha\mu)}$ and $r = 1 - \gamma\mu + \alpha\mu$.

r is a bifurcation parameter in the logistic model while μ is a bifurcation parameter in the SIS epidemic model. However, as stated above r and μ are related by $r = 1 - \gamma\mu + \alpha\mu$.

$0 < (\alpha - \gamma)\mu < 2$ provides the inequality $1 < 1 + (\alpha - \gamma)\mu < 3$, which corresponds to the condition $1 < r < 3$, which is the condition for asymptotic stability of a non-zero equilibrium point for the logistic model.

From [20], the ratio $\left(\mu_n - \mu_{n-1}\right)/\left(\mu_{n+1} - \mu_n\right)$ is equivalent to $\left(r_n - r_{n-1}\right)/\left(r_{n+1} - r_n\right)$, called the Myrberg or Feigenbaum number δ. From analysis [17] this ratio approaches a constant,

$$\delta = \lim_{n \to 0}\left(\mu_n - \mu_{n-1}\right)/\left(\mu_{n+1} - \mu_n\right) \approx 4.669202$$

since

$$\frac{\left(r_n - r_{n-1}\right)}{\left(r_{n+1} - r_n\right)} = \frac{\left(\left(1 + (\alpha - \gamma)\mu_n\right) - \left(1 + (\alpha - \gamma)\mu_{n-1}\right)\right)}{\left(\left(1 + (\alpha - \gamma)\mu_{n+1}\right) - \left(1 + (\alpha - \gamma)\mu_n\right)\right)} = \frac{\left(\mu_n - \mu_{n-1}\right)}{\left(\mu_{n+1} - \mu_n\right)}.$$

TABLE II
THE FEIGENBAUM CONSTANT

n	μ_n	$\mu_n - \mu_{n-1}$	$\left(\mu_n - \mu_{n-1}\right)/\left(\mu_{n+1} - \mu_n\right)$
1	0.740741		
2	0.907218	0.166477	4.751327
3	0.942256	0.035038	4.656831
4	0.94978	0.007524	4.66685
5	0.951392	0.001612	4.665595
6	0.951738	0.000346	4.688442
7	0.951811	7.37E-05	

Some numerical estimates of the Feigenbaum number are given in Table II. These estimates are, however, subject to appreciable numerical errors as the limit for δ approaches 0/0.

Fig. 6 shows how the solution behavior of (13) changes for $\alpha = 3.6$, $\gamma = 0.9$. For $\mu \in (0, 0.740741)$ the discrete equation has a stable equilibrium point, which corresponds with the stable equilibrium point of the continuous SIS model. For $\mu = 1$, the solution is chaotic.

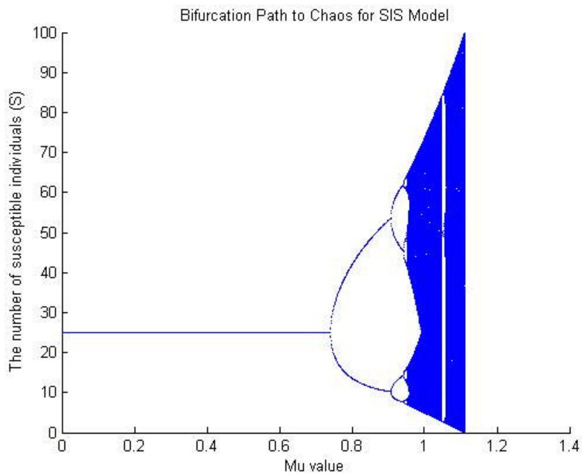

Fig. 6 The bifurcation diagram of μ. The parameter values in (13) are: $\alpha = 3.6$, $\gamma = 0.9$.

As shown in [17] (see also [21]), if there exists a period 3 cycle, then there exists chaotic behavior. For $\alpha = 3.6$, $\gamma = 0.9$, the bifurcation diagrams show a period three cycle for $\mu \in (1.0476, 1.0524)$ and also show chaos. More interesting behaviors are shown in Fig. 7 and Fig. 8.

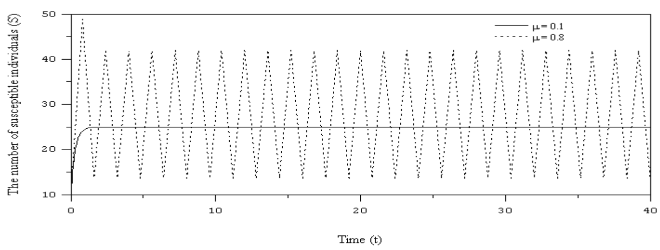

Fig. 7 The time series solution of (13) with $\alpha = 3.6$, $\gamma = 0.9$ and $N = 100$. The non-oscillatory solution occurs when $\mu = 0.1$ while oscillating period-2 solution occurs when $\mu = 0.8$.

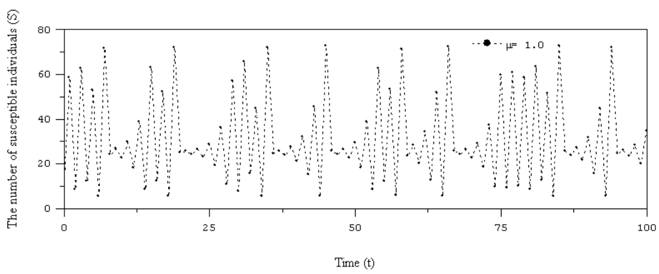

Fig. 8 The time series solution of (13) with $\alpha = 3.6$, $\gamma = 0.9$, and $N = 100$. The chaos occurs when $\mu = 1$.

To summarize, the (μ,α) parameter space is given in Fig. 9. It is delineated into three areas by the curves of $(\alpha-\gamma)\mu=2$ and $(\alpha-\gamma)\mu=\sqrt{6}$. The plots in Fig.9 are for parameter value $\gamma=0.9$. In area I, the solution of (13) is a single stable equilibrium state S_2^*. In area II, the solution of (13) is a stable period two cycle. In area III, the solutions of (13) are stable cycles of period 3 or more or correspond to chaos.

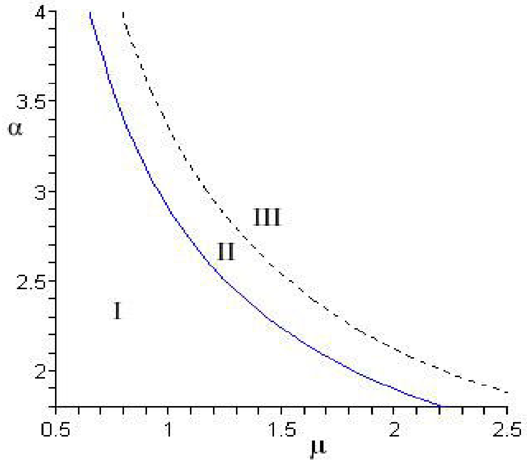

Fig. 9 The x-axis is μ and the y-axis is α. Area I is the region of stable equilibrium point, area II is the region of stable period two cycle, and area III is the region of stable higher period cycles and chaos. Parameter value $\gamma=0.9$.

VI. Conclusion and Discussion

The SIS epidemic model is well known in both continuous and discrete cases. Both of them give two distinct types of solution. The continuous case gives two equilibrium points which are asymptotically stable or unstable depending on parameter values. For the continuous case, stable oscillating solutions do not exist. On the other hand, the discrete case gives various types of oscillatory solutions such as period two cycle, period four cycle, period three cycle and also gives chaotic solutions in addition to the equilibrium point solutions of the continuous model. The time scales can produce large changes in the qualitative behavior of solutions.

Obviously, the continuous and discrete time SIS models behave sometimes differently, sometimes similarly depending on values of other parameters. It is clear that the choice of a time scales is important when building mathematical models and that predicted behaviors from a model can be qualitatively very different for different time scales. Therefore, in order to understand observed data in a variety of time measurements and predict its behavior precisely, choosing a suitable time scale is important for model formulation. Therefore, the theory of time scales can be a powerful tool for mathematical models.

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