On a finite horizon EOQ model with cycle dependent trade credit policies and time dependent parameters

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Abstract—In most inventory models it is assumed that the parameters of the model do not vary with time, and that the payment of orders from the retailer to the supplier is made immediately upon the receipt of these orders. Some suppliers, however, allow a certain fixed period to settle payment accounts. During this fixed period no interest is charged by the supplier, but beyond that period an interest, with the conditions agreed upon, is charged on the retailer. However, an interest can be earned by the retailer from the revenue that he receives during the given credit period. In this paper we develop and solve a general finite horizon trade credit economic ordering policy for an inventory model with deteriorating items where each of the demand, deterioration rates as well as all cost parameters are known and arbitrary functions of time. Both inflation and time value of money are incorporated in all cost components. The time horizon is divided into different cycles each of which has its own demand rate and its own trade credit period offered from the supplier to his retailer so that the retailer should pay his supplier before or after the end of the permissible trade credit of that cycle. Shortages are not allowed in any cycle. The objective of the retailer is to minimize his net total relevant costs. A closed form of this net total cost is derived and the resulting model is solved. Then rigorous mathematical methods are used to show that, under some seemingly possible conditions, there exists a unique vector of the relevant decision variables that solve the underlying inventory system.

Keywords—General EOQ; Variable trade credit Policies; Optimality; Deterioration; Inventory control; Time dependent parameters. Inflation, Time value of money

I. INTRODUCTION

It is implicitly known that retailers (buyers) pay their supplies (sellers) as soon as they receive the ordered items. In today's competitive environment of business, it is often that suppliers offer their retailers a certain trade credit period during which there is no charged interest so that the retailers have to settle their accounts by, at most, the end of the agreed upon credit period. Such offers provide the retailers with several advantages.

First, the retailers will have enough money to run their jobs. Second, retailers may benefit from the generated sales revenue by depositing it into an interest bearing account. Third, allowing the delay in payment may motivate the retailers to order more quantities which in turn lead to a reduction in the purchasing cost, ordering cost, and shortage cost. However, ordering large quantities will, in general, increase the holding cost, the cost of deteriorated or decaying of the stored items, and the potential cost due to inflation and/or time value of money. Therefore, the retailers are likely to balance between the effects of all above cost components so as to minimize the net total relevant cost.

The effect of supplier's trade credit policies on the traditional Economic Order Quantity (EOQ) has received the attention of many researchers. Goyal[12] has studied the effects of the permissible delay in payment on the standard EOQ model for non deteriorating items where shortages are not allowed. He showed that such delay in payment leads to an increase in both the order quantity and the replenishment interval and to a sharp decrease in the total annual cost. Aggarwal and Jaggi[1] presented a model similar to that of Goyal[12] but with constant deterioration rate where they presented a sensitivity analysis that reveal the effects of such deterioration rate on several factors of the EOQ policies. Jamal et al.[18] and Shah et al.[23] extended the models of Aggarwal and Jaggi[1] and Goyal[12], respectively, by allowing for shortages where it is further shown that the total cost is less than that in the non shortage case. Jaggi and Aggarwal[17] and Chung[9] considered a similar model as that of Aggarwal and Jaggi[1] but with a discount cash-flow (DCF) approach, and approximately reflected the effect of delay in payment on the optimal inventory policy. Kun et
al[16] considered the optimal ordering policy of the (EOQ) model under trade credit depending on the order quantity from the(DCF) approach. Some useful theoretical results on the subject of permissible delay in payment have been also achieved. Chung [10] provided conditions under which the total cost introduced by Goyal[12] is convex. Chu et al. [11] showed that the total cost of the system introduced by Aggarwal and Jaggi[1] is piecewise convex which in turn lead to an improved solution procedure of the considered system. Some other researchers have dealt with the subject from different point of view. Hwang and Shin [13] studied the case when the demand rate is a function of the retailer's price where it is, then, shown that the retailer's optimal price and the optimal lot size can be determined simultaneously. In Khouja and Mehrez [14] two main types of supplier credit policies, where the credit policy may be independent or linked to the order quantity, are addressed. Kim et al. [15] showed that the net profit of both the retailer and the supplies can be increased through a wise selection of the credit period. Balkhi [3] and [5] introduced a trade credit inventory model that generalizes most of the previously introduced models so that most of these models result as special case of Balkhi[3] model. Recently, other types of inventory control models with trade credits are proposed by many researchers. Luo [21] treated a single-vendor, a single-buyer supply chain for a single product, and a model to study and analyze the benefit of coordinating supply chain inventories through the use of credit period is proposed. Jinn-Tsair Teng[19] established an (EOQ) model for good retailers who receives a full trade credit by his supplier, and offers either a partial or a full trade credit to his bad customers. Sana. And Chaudhuri [22] modeled the retailer’s profit-maximizing strategy when confronted with supplier’s trade offer of credit and price-discount on the purchase of merchandise when retailers are facing many scenarios of time dependent demands for different kinds of goods. All above mentioned papers are of infinite time horizon type. A finite horizon inventory model with constant rate of deterioration and without shortages and with equal cycles, when a delay in payment is permissible, has been investigated by Liao et al. [20]. Balkhi[6] generalized the model of Liao et al [20] by considering that the given time horizon consists of different lengths each of which has its own demand rate and its own trade credit period. Most of the above mentioned papers considered that the cost parameters of the underlying inventory model as well as the demand and deterioration rates are all known and constant. Also, in most of those papers, neither inflation nor time value of money are taken in to accounts. Balkhi and Tadj [4] have studied a general EOQ model in which the cost parameters and both demand and deterioration rates are arbitrary functions of time. However, the effect of trade credit policies is not considered in this last model.

As it can be noted from the above literature review, there are several limitations while developing mathematical models in inventory control under trade credit policies. One of these limitations is that the assumption of constant demand and/or deterioration rates may not always be appropriate for many inventory items as, for example, it is the case for products whose demand is affected by seasons or occasions. Another limitation is the case of a given finite horizon when it is divided it into equal inventory cycles. A third one is that the permissible delay in payment (i.e., the trade credit period) is assumed to be the same for all cycles. A fourth limitation is concerned with constant cost parameters which is not realistic for most practice inventory control systems. Thus, there is a need to drop the above limitations and develop a more general inventory model in which the demand and deterioration rates are functions of time, each cycle has its own demand rate and its own trade credit so that many of the previously introduced models can be obtained (if needed) as special cases of a such general model.

The purpose of this paper is to generalize the paper of Balkhi [2] and most of the above introduced models in the following fronts.

(i) The given time horizon consists of \( n \) different cycles each of which has its own trade credit period, and its own ordered quantity which is equal to the inventory level at the start of that cycle.

(ii) Both demand and deterioration rates are general and continuous functions of time. Further the demand rate is cycle dependent which allows the possibility to choose the demand to be stock dependent. Therefore our order quantity is cycle and time dependent by assumption (i) (ii)

(iii) All cost parameters are time Dependent
The two possibilities, that the credit period may be less or greater than the cycle length, are conventionally incorporated for each cycle so that the proposed model can suite more practical and possible cases, including the cases of prepayments, immediate payments, and late payments.

(v) The effects of both inflation and time value of money are incorporated in all cost components. Our main concern will be on the theoretical results. So, the proposed model with the above mentioned general features is developed, solved, and no approximations are used neither in the total net cost nor in any other relations. Then, rigorous mathematical methods are used to prove the existence of a unique vector of the relevant decision variables that solve the underlying inventory system.

II. Model Assumptions and Notation

(1) A single item is ordered at the beginning of the first cycle and held in stock in a predetermined period of \(H\) units long.

(2) The time horizon is divided into \(n\) different cycles.

(3) For cycle \(j (j = 1, 2, \ldots, n)\) we denote by

\[ D_j(t) \text{ the demand rate at time } t. \]

\[ M_j \text{ the trade credit period offered from the supplier to his retailer so that the retailer should pay his supplier before or after the end of this permissible trade credit of that cycle.} \]

(4) The ordered items deteriorate while are effectively in stock and the deterioration rates is an arbitrary and known function of times denoted by \(\theta(t)\). But there is no repair or replacement of deteriorated items.

(5) \(t_{j-1}\) is the beginning of the cycle \(j\) and \(t_j\) is the end of the cycle with \(t_0 = 0\) and \(t_n = H\).

(6) \(Q_j\) is the quantity ordered at the beginning of the cycle, and \(I_j(0)\) is the inventory level at time \(t\).

(7) Lead time is negligible and the replenishment rate is infinite.

(8) Shortages are not allowed in any cycle.

(9) A complete order of \(Q_j\) units arrives by the beginning of the cycle, but there is a permissible delay for payment up to the end of a period of length \(M_j\) (which is usually called trade credit period) given by the supplies (or whole sailor) to the buyer (or retailer) for settling his account in that cycle.

(10) The cost structure for any cycle of the underlying inventory system is as follows:

\[ c(t) \text{ is the unit cost at time } t. \]

\[ h(t) \text{ is the holding cost per unit per unit time at time } t. \]

\[ s(t) \text{ is the unit selling price at time } t. \]

\[ k(t) \text{ is the ordering cost per order at time } t. \]

All above costs are free of interest charges.

(11) All costs are affected by inflation rate and time value of money. We shall denote by \(r_1\) the inflation rate and by \(r_2\) the discount rate representing the time value of money so that \(r = r_2 - r_1\) is the discount rate of inflation.

There is an interest charged for the items being held in stock after the given trade credit period \(M_j\) (i.e., for those items not being sold during \(M_j\)). We denote by \(i_e\) the per monetary unit per unit time charge payable at time \(t = 0\).

(12) All generated sales revenue are deposited into an interest bearing account at a rate \(i_e\) (at time \(t = 0\)) per monetary unit per unit time.

We generally have \(i_e > i_e\).

The proposed inventory trade credit system operates as follows. A quantity of \(Q_j\) units is ordered and stored at the beginning of cycle \(j\). The supplier (or seller) offers the buyer (or retailer) a certain trade credit period \(M_j\) during which there is no charged interest so that the account is to be settled by at most the end of the given permissible credit period. Otherwise an interest of \(i_e\) will be charged for any amount delayed beyond that allowed period. On the other hand the retailer may deposit all generated sales revenue into an interest bearing account of \(i_e\) and may pay off his supplier gradually. Whereas the inventory system operates as follows. A quantity of \(Q_j\) units is ordered at the beginning of cycle \(j\). This is subject to consumption of rate \(D_j(t)\) and deterioration of rate \(\theta(t)\) till the inventory level reach to zero by time \(t_j\) where a new order for the next cycle is ordered. The objective of the retailer is to minimize his net total relevant costs in any cycle. The process is repeated. Fig 1. show the variation of the underlying inventory system.

III. Model Building

Following the above assumptions and notations, the changes of \(I_j(t)\) in any cycle \(j\) are given by the following differential equation

\[
\frac{dI_j(t)}{dt} = -D_j(t) - \theta(t)I_j(t) \quad t_{j-1} \leq t \leq t_j
\]

with the boundary condition \(I(t_j) = 0\). The solution of (1) is given by:
The variation of the inventory system in a finite horizon of $H$ time units with different cycles

$$\int_0^H \delta(u) \delta(t) k \ dt \ du$$

(2)

where,

$$\delta(t) = \int_0^t \delta(\theta) \ d\theta$$

(3)

Next we derive the present worth of all cost components for cycle $j$ ($j=1,2,...,n$).

The present worth of the holding cost (PWHC) of the amount being held in stock during the period $[t_{j-1}, t_j]$ is given by

$$PWHC = \int_{t_{j-1}}^{t_j} e^{-rt} h(t) I_j(t) dt$$

integrating by parts, the last is reduced to:

$$PWHC = \int_{t_{j-1}}^{t_j} e^{-rt} \left( \int_{t_{j-1}}^{t_j} e^{rt} D_j(u) du \right) dt$$

Integrating by parts, the last is reduced to:

$$PWHC = \int_{t_{j-1}}^{t_j} \left( H(t) - H(t_{j-1}) \right) e^{rt} D_j(t) dt$$

(4)

where,

$$H(t) = \int_0^t e^{-rt} h(t) dt$$

(5)

The number of items received in beginning of the cycle is given by

$$Q_j = I(t_{j-1}) = e^{rt_{j-1}} \int_{t_{j-1}}^{t_j} e^{rt} D_j(u) du$$

(6)

Since items are received at the beginning on each cycle (recall that lead time is negligible), the unit item's cost is equal to $c(t_{j-1})$ and the ordering cost is equal to $k(t_{j-1})$.

Hence, The present worth of the items' cost (PWIC) is equal to

$$PWIC = c(t_{j-1}) Q_j = c(t_{j-1}) e^{-rt_{j-1}} \int_{t_{j-1}}^{t_j} e^{rt} D_j(u) du$$

(7)

Note that the items' cost include the cost of deteriorated and consumed (non-deteriorated) items.

The present worth of ordering cost (PWOC) is equal to

$$e^{-rt_{j-1}} k(t_{j-1})$$

To find the present worth for each of the interest charged and the interest earned we distinguish two cases.

Case (1). Cycles with $M_j \leq t_j$.

In this case we need to find the present worth of the cost which will result from interest charged for items in inventory not being sold after $M_j$. But the number of items not being sold in a small time interval $dt$ after $M_j$ is equal to $I_j(t) dt$. Thus the present worth of interest charged (PWIG) is given by

$$PWIG = i_c \int_{M_j}^{t_j} e^{-rt} c(t) I_j(t) dt$$

Integrating by parts, then the last amount is equal to

$$PWIG = i_c \int_{M_j}^{t_j} \left[ G(t) - G(M_j) \right] e^{rt} D_j(t) dt$$

where ,

$$G(t) = \int_0^t e^{-rt} c(t) dt$$

(9)

Similarly, the present worth of the interest earned (PWIE) in the permissible period $[t_{j-1}, M_j]$, which has positive stock, and in the rest period $[M_j, t_j]$ from the remaining cash, is given by:

$$PWIE = i_c \int_{M_j}^{t_j} e^{-rt} s(t) D_j(t) dt +$$

$$i_c \int_{M_j}^{t_j} e^{-rt} s(t) D_j(t) dt$$

(10)

Thus the net total variable cost in cycle $j$ is given by
\[ W_{ij}(t_{j-1}, t_j) = \int_{t_{j-1}}^{t_j} [H(t) - H(t_{j-1})] e^{\delta(t)} D_j(t) \, dt + \]
\[ c(t_{j-1}) e^{\gamma(t_j - t_{j-1})} \int_{t_{j-1}}^{t_j} e^{\delta(u)} D_j(u) \, du + k(t_{j-1}) e^{\gamma(t_j - t_{j-1})} - \]
\[ i_c \int_{t_{j-1}}^{t_j} e^{-\gamma(s)} s(t) D_j(t) \, dt \]

\[ W_j(t_{j-1}, t_j) = \int_{t_{j-1}}^{t_j} [H(t) - H(t_{j-1})] e^{\delta(t)} D_j(t) \, dt + \]
\[ + c(t_{j-1}) e^{\gamma(t_j - t_{j-1})} \int_{t_{j-1}}^{t_j} e^{\delta(u)} D_j(u) \, du + k(t_{j-1}) e^{\gamma(t_j - t_{j-1})} - \]
\[ + \alpha_j \int_{t_{j-1}}^{t_j} G(t) e^{\delta(t)} D_j(t) \, dt \]
\[ - i_c \int_{t_{j-1}}^{t_j} e^{-\gamma(s)} s(t) D_j(t) \, dt \]

\[ i_c (1 - \alpha_j)(M_j - t_j) \int_{t_j}^{t_j} e^{-\gamma(s)} s(t) D_j(t) \, dt \]

**Case (2). Cycles with** \( M_j \geq t_j \).

In this case we do not have an interest charge for inventory not being sold after \( M_j \) since the inventory is exhausted by time \( t_j \). But the interest earned per cycle consists, here, from two parts. The first part is the interest earned during the positive inventory period \([ t_{j-1}, t_j ]\) which is given by

\[ i_c \int_{t_{j-1}}^{t_j} e^{-\gamma(s)} s(t) D_j(t) \, dt \]

The second part is the interest earned from the remaining cash during the time period \([ t_j, M_j ]\) after the depletion of inventory. The present worth of this last part is equal to

\[ i_c (M_j - t_j) \int_{t_j}^{M_j} e^{-\gamma(s)} s(t) D_j(t) \, dt \]

Thus the net total variable cost in cycle \( j \) in Case (2) as a function of \( t_{j-1} \) and \( t_j \), say \( W_j(t_{j-1}, t_j) \), is given by

\[ W_j(t_{j-1}, t_j) = \int_{t_{j-1}}^{t_j} [H(t) - H(t_{j-1})] e^{\delta(t)} D_j(t) \, dt + \]
\[ c(t_{j-1}) e^{\gamma(t_j - t_{j-1})} \int_{t_{j-1}}^{t_j} e^{\delta(u)} D_j(u) \, du + k(t_{j-1}) e^{\gamma(t_j - t_{j-1})} - \]
\[ i_c \int_{t_{j-1}}^{t_j} e^{-\gamma(s)} s(t) D_j(t) \, dt - i_c (M_j - t_j) \int_{t_j}^{M_j} e^{-\gamma(s)} s(t) D_j(t) \, dt \]

Now, define \( \alpha_j \) by

\[ \alpha_j = \begin{cases} 1 & \text{if } M_j < t_j \text{ in cycle } j \\ 0 & \text{Otherwise} \end{cases} \]

Then we can unify (13) and (14) by the following formula

\[ W_j(t_{j-1}, t_j) = \int_{t_{j-1}}^{t_j} [H(t) - H(t_{j-1})] e^{\delta(t)} D_j(t) \, dt + \]
\[ + c(t_{j-1}) e^{\gamma(t_j - t_{j-1})} \int_{t_{j-1}}^{t_j} e^{\delta(u)} D_j(u) \, du + k(t_{j-1}) e^{\gamma(t_j - t_{j-1})} - \]
\[ + \alpha_j \int_{t_{j-1}}^{t_j} G(t) e^{\delta(t)} D_j(t) \, dt \]
\[ - i_c \int_{t_{j-1}}^{t_j} e^{-\gamma(s)} s(t) D_j(t) \, dt - i_c (1 - \alpha_j)(M_j - t_j) \int_{t_j}^{M_j} e^{-\gamma(s)} s(t) D_j(t) \, dt \]

where \( t_0 = 0 \) and \( t_n = H \). Thus our problem, which we shall refer to as \((P)\), is

Minimize \( W \) given by (18)

Subject to:

\[ 0 = t_0 \leq t_1 \leq t_2 \leq \ldots \ldots \leq t_n = H \]

Note that constraints (19) must be satisfied for any feasible solution of the underlying problem since otherwise the problem would have no meaning.

**IV. Problems' Solution and its Minimality**
To solve problem (P) we shall first ignore constraints (19) and consider \( n \) to be fixed. We refer to the resulting unconstrained problem as \((P_u)\).

Then, the necessary conditions for having a minimizing solution for \((P_u)\) are:

\[
\frac{\partial W}{\partial t_j} = 0 \quad , j=1,2,\ldots, n-1 \tag{20}
\]

Denoting by \( Y'(x) = \partial Y / \partial x \) then

\[
(20) \Leftrightarrow \sum_{j=1}^{n-1} \left\{ [c(t_j)(r+\theta(t_j))-c'(t_j)] + h(t_j) \right\} e^{-\gamma t_j - \delta(t_j)} \times \int_{t_j}^{t_{j+1}} e^{\delta(t)} D_{j+1}(t)dt \}
\]

\[
= \sum_{j=1}^{n-1} \left\{ i_c ((1-\alpha_j)(M_j-t_j) - \int \right\} e^{\gamma t_j} s(t_j) D_j(t) dt + \int_{t_j}^{t_{j+1}} e^{\delta(t)} D_{j+1}(t) dt
\]

\[
\{ [H(t_j) - H(t_{j+1})] + c(t_{j+1}) e^{\gamma t_{j+1} - \delta(t_{j+1})} + \alpha_j \int_{t_j}^{t_{j+1}} e^{\gamma t_j} s(t_j) D_j(t_j) dt + \int_{t_j}^{t_{j+1}} e^{\delta(t)} D_{j+1}(t_j) dt + [k(t_j) - rk(t_j)] e^{\gamma t_j} \} : j=1,2,\ldots n-1 \tag{21}
\]

Note that (21) are \((n-1)\) equations with \((n-1)\) decision variable, namely \( t_1 \), \( t_2 \), \ldots, \( t_{n-1} \). The solution of these equations (if it exists) gives the critical points of \( W \). Now, let

\[
W_{jj} = \frac{\partial^2 W}{\partial t_j} \quad W_{jk} = \frac{\partial^3 W}{\partial t_j \partial t_k} \tag{22}
\]

we then have

\[
W_{j(j+1)} = -[c(t_j)(r+\theta(t_j)) - c'(t_j) + h(t_j)] e^{\gamma t_j - \delta(t_j)} D_{j+1}(t_{j+1}), \quad j=1,2,\ldots n-1 \tag{23}
\]

\[
W_{j(j+1)} = c(t_{j+1}) e^{\gamma t_j - \delta(t_j)} \times e^{\delta(t_j)} D_j(t_j)
\]

\[
-c(t_{j+1})(r+\theta(t_{j+1})) e^{\gamma t_{j+1} - \delta(t_{j+1})} \times e^{\delta(t_{j+1})} D_j(t_j)
\]

\[
j = 2,3,\ldots ,n \tag{24}
\]

\[
W_{jk} = 0 \quad \text{for} \quad j \geq k+2 \quad \text{or} \quad j \leq k-2 ,
\]

\[
W_{jj} = \left( r+\theta(t_j) \right) [c(t_j)(r+\theta(t_j)) - c'(t_j) + h(t_j)] e^{\gamma t_j - \delta(t_j)} \times \int_{t_j}^{t_{j+1}} e^{\delta(t)} D_{j+1}(t) dt + \int_{t_j}^{t_{j+1}} e^{\delta(t)} D_{j+1}(t) dt +
\]

\[
-c'(t_j)(r+\theta(t_j)) + c(t_j) \theta(t_j) - c''(t_j) + h'(t_j) e^{\gamma t_j - \delta(t_j)} \times \int_{t_j}^{t_{j+1}} e^{\delta(t)} D_{j+1}(t) dt +
\]

\[
[c(t_j)(r+\theta(t_j)) - c'(t_j) + h(t_j)] e^{\gamma t_j - \delta(t_j)} D_{j+1}(t_{j+1}) - i_c \int_{t_j}^{t_{j+1}} e^{\delta(t)} s(t_j) D_j(t_j) dt - i_c \int_{t_j}^{t_{j+1}} e^{\delta(t)} s(t_j) D_j(t_j) dt + [H(t_j) - H(t_{j+1})] c(t_{j+1}) e^{\gamma t_{j+1} - \delta(t_{j+1})} + \alpha_j i_c \int_{t_j}^{t_{j+1}} e^{\delta(t)} D_j(t_j) dt + [H(t_j) - H(t_{j+1})] + c(t_{j+1}) e^{\gamma t_j - \delta(t_j)} + \alpha_j i_c \int_{t_j}^{t_{j+1}} e^{\delta(t)} D_j(t_j) dt - r i_c \left[ s(t_j) - c(t_j) \right] e^{\gamma t_j} D_{j+1}(t_j) - r i_c \left[ s(t_j) - c(t_j) \right] e^{\gamma t_j} D_{j+1}(t_j) - r [k(t_j) - rk(t_j)] e^{\gamma t_j} \} : j=1,2,\ldots n-1 \tag{25}
\]

Next we deliver sufficient conditions for which any existing solution of \((P_u)\) is a minimizing solution to \((P)\). For this purpose, let \( T = (0=t_0, t_1, t_2, \ldots , t_n=H) \) be a solution of equations (21) and let \( HM(T) \) be the value of the Hessian Matrix at \( T \), then, by Bazara et al [8], Stewart [24, page 143 Chapter 3] and Theorem 3 of Balkhi and Benkhrouf [7], and Balkhi [6], \( HM(T) \) is positive definite if

\[
W_{jj} > \left| W_{j(j+1)} + W_{j(j-1)} \right| \quad j = 2, \ldots, n-2 ,
\]

\[
W_{jj} > \left| W_{j(j+1)} \right| \quad \text{for} \quad j = 1, \text{and}
\]

\[
W_{jj} > \left| W_{j(j-1)} \right| \quad \text{for} \quad j = n-1 \tag{26}
\]

Thus, the above arguments lead to the following theorem.

**Theorem 1.** For fixed \( n \), any existing solution of \((P)\) is a minimizing solution to \((P)_u\) if this solution satisfies conditions (26).

**IV. Uniqueness and Global Optimality of the Solution to \((P)_u\)**
In this section we shall show that any existing and minimizing solution of \((P_j)\) is unique (hence global optimal) of both \((P_j)\) and \((P)\). We shall show this in three steps:

First we show that \((P_j)\) depend only on one of the variables \(t_1, t_2, ... , t_{n-1}\). Second, we showed that under the hypothesis of Theorem 1, then any existing and minimizing solution of \((P_j)\) is unique. Third, we showed that under the hypothesis of Theorem 1, the total net relevant cost \(W\) is convex with respect to \(n\). Now, from equations \((21)\) and for \(j=1\) we have

\[
[c(t_j)(r + \theta(t_j)) - c'(t_j) + h(t_j)]e^{-\gamma_j}\delta(t_j)\int_0^t e^{\delta(t)}D_2(t)dt \]

\[
[e^{\alpha_j}\int_0^t e^{\delta(t)}D_2(t)dt + H(t_j) + H(t_0) + c(t_0) \alpha_j + \int_0^t (\alpha_j + \beta_j) e^{\delta(t)}D_2(t)dt + \int_0^t \alpha_j \int_0^t e^{\delta(t)}D_2(t)dt + \int_0^t \int_0^t e^{\delta(t)}D_2(t)dt + [K(t_0) - rk(t_0)]e^{\gamma_j}]
\]

which leads, from \((28)\), to

\[
W_j(t_j) t_j > t_j' > t_j' > t_j' > W_j(t_j)
\]

from which the desired result follows.

Now relations \((28)\) lead to the following corollary

**Corollary 1.** Under the hypothesis of Theorem 1, then,

(i) All variables \(t_1, t_2, ... , t_{n-1}\) are increasing functions of \(t_1\) and of each others.

(ii) For fixed \(n\), any existing and minimizing solution to problem \((P)\) is an existing and minimizing solution to problem \((P)\).

**Proof.** The proof of part (i) is clear from relations \((28)\). Again, from \((28)\) and by the theory of Real Analysis we have

\[
t_j \geq t_{j-1} \geq 0, j = 1, 2, ..., n-1
\]

Thus constraints \((19)\) hold for any existing and minimizing solution of \((P)\) if \(0 \leq t_j\). But, as an implication of Kohn-Tucker necessary conditions, the last inequality needs not to be considered. Hence, constraints \((19)\) are satisfied for any feasible solution of \((21)\) and hence such a solution is an existing and minimizing solution to \((P)\).

Some other main results follow.

**Theorem 2.** Under the hypothesis of Theorem 1, any existing and minimizing of \((P)\) is the unique solution of \((P)\).

**Proof.** Let \(T = (0 = t_0, t_1, t_2, ... , t_n = H)\) be an existing and minimizing of \((P)\) . By relations \((27)\) and \((30)\), the amount \(\sum_1^n (t_j - t_{j-1}) - H\) is a function of \(t_1\). Our idea is to show that the equation

\[
\sum_1^n (t_j - t_{j-1}) - H = 0
\]

as an equation of \(t_1\) either has a unique solution or it does not have any solution. To see this, let us denote by \(Z(t_1)\) the left hand side of \((31)\) then

\[
Z'(t_1) = \sum_1^n (t_j' - t_j') > 0 \text{ by } (28), \text{ which means that } Z(t_1) \text{ is an increasing function of } t_1.
\]

Now, if \(Z(0) \leq 0\) then \((31)\) has a unique solution, if however \(Z(0) > 0\) then \((31)\) does not have any solution (see Fig. 2). This leads to the desired result.

**Theorem 3.** Consider the following two different existing schedules with \(n\) and
If the hypothesis of theorem 1 hold , then the entries of $\hat{S}$ lies between the entries of $S$, that is;

$$0 = \hat{t}_0 \leq \hat{t}_1 \leq \hat{t}_2 \leq \ldots \leq \hat{t}_n = \hat{t}_{n+1} = H$$

the two replenishment schedules $S=(T,n)$, $\hat{S}=(\hat{T},n+1)$ then we have that $\hat{t}_i \leq t_i$ by Theorem 3. Hence $W_R(\hat{S}) \leq W_R(S)$. This means that $W_R$ is an increasing function of $t_i$ but a non increasing function of $n$. On the other hand we have $W_0 = \sum_{j=1}^{n+1} k(t_{ji}) e^{rt_i} > k(0) > 0$. It is clear that $W_0$ is an increasing function of $n$. Now, combining the above results we can reach to the following conclusion. While $W_R$ decreases with $n$, $W_0$ increases with $n$ so that $W_0$ will eventually dominate $W_R$ after certain value of $n$ say $n^*$ where $W = W_0 + W_R$ starts to increase with $n$. Hence there is a unique vector $S^* = (T^*,n^*)$ that solves the underlying inventory problem. This completes the proof of the theorem.

**Proof.** The proof of this theorem mimics the proof of Theorem 3 of Balkhi[6] so it will not be repeated again

**Theorem 4.** Suppose that the hypothesis of Theorem 1 hold and $MS$ is the set of all minimizing solutions of the form $S=(T,n)$ for the underlying inventory system , then there exists a unique vector $S^* = (T^*,n^*)$ from $MS$ for which the net total relevant cost of this system is minimum if the following condition holds

$$c^*(t_i) - c(t_i)(r + \theta(t_i)) - h(t_i) \geq 0 \quad (32)$$

**Proof.** From (18) we can rewrite $W$ as the sum of set up costs and the rest of cost components, that is;

$$W = W_0 + W_R$$

where $W_0 = \sum_{j=1}^{n} k(t_{ji}) e^{rt_i}$ and

$$W_R = W - W_0, \text{ then from (21) we have}$$

$$\frac{\partial W^2}{\partial t_i} = i_c(1 - \alpha_i)(M_i - t_i) \int_{t_i}^{M_i} e^{ct_i(s)} D_i(s) ds dt +$$

$$[H(t_i) - H(t_0)] + c(t_i) e^{\gamma t_i} [\alpha_i i_c (G(t_i) - G(M_i))] e^{\delta t_i} D_i(t_i) +$$

$$[i_c(1 - \alpha_i)(M_i - t_i) - 1] e^{\gamma t_i} S(t_i) D_i(t_i) +$$

$$i_c[ s(t_i) - c(t_i) ] e^{\delta t_i} D_i(t_i) +$$

$$\{ [c^*(t_i) - c(t_i)(r + \theta(t_i)) - h(t_i)] e^{rt_i} D_i(t_i) \} \times$$

$$\int_{t_i}^{M_i} e^{\gamma(s)} D_i(s) ds > 0$$

if (32) holds. This means that $W_R$ is an increasing function of $t_i$ by (32). Now, consider

**V. Conclusion**

In this paper we have presented a general economic order quantity (EOQ) inventory model for deteriorating items under inflation and time value of money. Both demand and deterioration rates as well as all cost parameters are general and continuous functions of time. The given time horizon consists of "n" different cycles each of which has its own demand rate and its own trade credit period which is usually offered from the supplier to the retailer. In contrast with the previous literature, some of these credit periods are taken to be less than the end of the corresponding cycles and the rest are taken to be greater than or equal to such ends. The retailer receives the items needed for any cycle at the beginning of this cycle, but he may delay the payment to his supplier up to the end of the credit period concerned with that cycle without bearing any interest. On the other hand, the retailer can deposit all sales revenue of any cycle at an interest bearing account and uses the aggregated amounts of revenue to pay his supplier by, at most, the end of the permissible credit period of the cycle. However, after the credit period of any cycle an interest is charged for unsold items. The main objective of the retailer is to minimize the net total relevant cost which consists of the sum of the charged interest of unsold items, the items cost, the holding cost and the ordering cost minus the interest earned from the generated sales.
A closed formula of the net total cost is derived in terms of the cost parameters and the relevant decision variables which are the replenishment points and the number "n" of the cycles. Solution procedures for the resulting model are developed without any approximation neither for the net total relevant cost nor for any other mathematical relation. Then, sufficient conditions which lead to minimizing solution(s) for each given \( n \) and for any existing solution(s) are derived. Rigorous mathematical methods are used to prove that, from all minimizing solutions, there is a unique global optimal solution in terms of \( n \) and the replenishment points. Several previous models are special cases of the proposed models. This seems to be the first time where such a general trade credit inventory model is theoretically investigated, solved, and its optimality conditions are established.

REFERENCES


