Order Ideals of a Quasi-ordered Set and Graywater

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Abstract: Reuse of graywater is often suggested to be among the most effective to some of the current ecological problems. Application of rough sets theory for discovering dependences between various types of graywater is in the main focus of this work. This can facilitate processes aiming at less energy and chemical use and lower fresh water use.

Key–Words: Rough sets, order ideals, quasi-ordered set

1 Introduction

Ship generated waste streams

Graywater, bilge water, sewage, ballast water, and solid waste are among the most discussed ship generated waste streams. Their potential environmental impacts are seriously considered by the scientific community.

Bilge water is the one that collects in the lowest part of the ships hull and may contain oil, grease, and other contaminants. Graywater is waste water from showers, sinks, laundries and kitchens and ballast water is the one taken onboard or discharged from a vessel to maintain its stability.

Some ships discharge graywater untreated while others practice limited treatment like gross particle filters or grease traps.

In this work we apply rough sets approximation for relating different types of graywater. The outcome can be used for facilitating an automated graywater treatment process based on graywater sources.

The rest of the paper is organized as follows. Section 2 contains definitions of terms used later on. Rough sets approximation of graywater are discussed in Section 3. Section 4 contains the conclusion of this work.

2 Lattices and Rough Sets

2.1 Lattices

Let $P$ be a non-empty ordered set. If $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for all $x, y \in P$, then $P$ is called a lattice [3]. In a lattice illustrating partial ordering of knowledge values, the logical conjunction is identified with the meet operation and the logical disjunction with the join operation. A partially ordered set in which all subsets have both a supremum (join) and an infimum (meet) is a complete lattice.

Let $L$ and $K$ be lattices. Define $\vee$ and $\wedge$ coordinate wise as follows

$$(l_1, k_1) \vee (l_2, k_2) = (l_1 \lor l_2, k_1 \lor k_2),$$

$$(l_1, k_1) \wedge (l_2, k_2) = (l_1 \land l_2, k_1 \land k_2).$$

The lattice formed by taking the ordered set product of the lattices $L$ and $K$ is the same as the one obtained by defining $\vee$ and $\wedge$ coordinate wise on the product $L \times K$, [3].

2.2 Concept Lattices

A context is a triple $(G, M, I)$ where $G$ and $M$ are sets and $I \subset G \times M$. The elements of $G$ and $M$ are called objects and attributes respectively [3], [5].

For $A \subseteq G$ and $B \subseteq M$, define

$$A' = \{m \in M \mid (\forall g \in A) gIm\},$$

$$B' = \{g \in G \mid (\forall m \in B) gIm\}$$

where $A'$ is the set of attributes common to all the objects in $A$ and $B'$ is the set of objects possessing the attributes in $B$.

A concept of the context $(G, M, I)$ is defined to be a pair $(A, B)$ where $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. The extent of the concept $(A, B)$ is $A$ while its intent is $B$. A subset $A$ of $G$ is the extent of some concept if and only if $A'' = A$ in which case the unique concept of which $A$ is an extent is $(A, A')$.
The corresponding statement applies to those subsets \( B \in M \) which are the intents of some concepts.

The set of all concepts of the context \((G, M, I)\) is denoted by \( \mathfrak{B}(G, M, I) \). \( (\mathfrak{B}(G, M, I), \subseteq) \) is a complete lattice and it is known as the concept lattice of the context \((G, M, I)\).

The context sum in formal concept analysis corresponds to the direct product of the two concept lattices.

The sum of the formal contexts \((G, M, I)\) and \((H, N, J)\) where \( G \cap H = \emptyset = M \cap N \) is the formal context
\[
(G \cup H, M \cup N, I \cup J \cup (G \times N) \cup (H \times M)).
\]

### 2.3 Distributive Lattices

A bounded distributive lattice is a structure \( L = (L, \vee, 0, \wedge, 1) \) such that \((L, \vee, \wedge)\) is a distributive lattice, 0 is the least element: \( 0 \leq x \), and 1 is the greatest element: \( x \leq 1 \).

A distributive \( p\)-algebra is a structure \( L = (L, \vee, 0, \wedge, 1, \ast) \) such that \( L = (L, \vee, 0, \wedge, 1) \) is a bounded distributive lattice, \( x^\ast \) is the pseudo complement of \( x \): \( y \leq x^\ast \iff x \wedge y = 0 \).

A Stone algebra is a distributive \( p\)-algebra \( L = (L, \vee, 0, \wedge, 1, \ast) \) such that \( (x^\ast)^\ast \vee x^\ast = 1 \) and \( 0^\ast = 1 \).

The interior operator \( ^\circ \) is dual to the closure operator \( ^\circ \), in the sense that \( S^\circ = X \setminus (X \setminus S)^\circ \), and also \( S^\circ = X \setminus (X \setminus S)^\circ \) where \( X \) is the topological space containing \( S \), and the backslash refers to the set-theoretic difference.

A subalgebra of an algebra over a commutative ring or field is a vector subspace which is closed under the multiplication of vectors.

On the Cartesian product of two sets with binary relations \( R \) and \( S \), define \((a, b)T(c, d)\) as \( a \sim c \) and \( bSd \). If \( R \) and \( S \) are both reflexive, irreflexive, transitive, symmetric, or antisymmetric, relation \( T \) has the same property.

### 2.4 Rough Sets

According to the theory of rough sets, a set belongs to a universe \( U \), where elements of \( U \) can be specified only up to an indiscernibility equivalence relation \( \sim \) on \( U \). If a subset \( A \subseteq U \) is rough, if \( A \) contains an element indiscernible from another element not in \( A \). A rough set \( A \) is usually described by two approximations:

the upper approximation
\[
\overline{R}(A) := \{ u \in U \mid \exists y \sim u \ y \in A \}, \text{ and}
\]

the lower approximation
\[
\underline{R}(A) := \{ u \in U \mid \forall y \sim u \ y \in A \}.
\]

<table>
<thead>
<tr>
<th>Table 1: A bond from ((G, M, I)) to ((H, N, J))</th>
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<th>Table 2: A bond from ((G, M, I)) to ((H, N, J))</th>
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<td>( H )</td>
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Let \( U \) be a set, let \( X \rightarrow \overline{R}(X) \) be an interior operator and \( X \rightarrow \overline{R}(X) \) be a closure operator on \( U \). The complete sublattice generated by the pairs \((\overline{R}(X), \overline{R}(X))\) is called the lattice of rough set abstractions.

A quasi-order, i.e., a reflexive and transitive relation.

Let \( P \) be a fixed set. A \( P\)-lattice \((L, \alpha)\) consists of a complete lattice \( L \) together with a mapping \( \alpha : P \rightarrow L \), the image of which generates \( L \). If \((L_1, \alpha_1)\) and \((L_2, \alpha_2)\) are \( P\)-lattices, then their \( P\)-product is the complete sublattice of \( L_1 \times L_2 \) that is generated by the pairs

\[
\{ \alpha_1(p), \alpha_2(p) \mid p \in P \}.
\]

A \( P\)-product automatically is a subdirect product, because each component contains a generating set of the respective factor. \( P\)-products of concept lattices correspond to \( P\)-fusions of their contexts. We briefly sketch the construction here; details can be found in [5]. \( P\)-fusions are built from bonds.

Let \((G, M, I)\) and \((H, N, J)\) be formal contexts. A bond from \((G, M, I)\) to \((H, N, J)\) is a relation \( R \subseteq G \times N \) with the property that
\[
g^R \text{ is an intent of } (H, N, J) \text{ for every } g \in G, \text{ and}
\]
\[
n^R \text{ is an extent of } (G, M, I) \text{ for every } n \in N, \text{ Table 1.}
\]

The \( P\)-fusion of the two \( P\)-contexts \((G, M, I, \alpha_1)\) and \((H, N, J, \alpha_2)\) is defined as the formal context

where for \( \{i, j\} = \{1, 2\} \) the relation \( R_{i,j}^3 \) is the smallest bond containing
\[
R_{i,j} := \bigcup_{p \in P} A_{i,p}^p \times A_{j,p}^p.
\]
Table 3: Formal context for the lattice of rough set approximations.

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Theorem 1 [5] Let \(((G, M, I), \alpha_1)\) and \(((H, N, J), \alpha_2)\) be \(P\)-contexts. The concept lattice of their \(P\)-fusion is isomorphic to the \(P\)-product of their concept lattices.

The formal concepts of the \(P\)-fusion are precisely the pairs \((A_1 \cup A_2, B_1 \cup B_2)\), where \((A_1, B_1)\) and \((A_2, B_2)\) are formal concepts of \((G, M, I)\) and of \((H, N, J)\), respectively, such that the pair \(((A_1, B_1), (A_2, B_2))\) is an element of the \(P\)-product.

Theorem 2 [6] Let \(\leq\) be a quasi-order on the universe \(U\), having no isolated points. For \(X \subseteq U\) define

\[
\overline{R}(X) := \{ u \in U | \exists x \geq u x \in X \}
\]

\[
\underline{R}(X) := \{ u \in U | \exists x \leq u x \in X \}.
\]

Then the lattice of rough set approximations equals the order relation of the lattice of order ideals of the quasi-ordered set \((U, \leq)\).

The rough set approximations form a sublattice of its square, and this lattice is distributive as well, [6].

3 Lattice of Rough Set Approximations

A formal context for the lattice of rough set approximations for objects: graywater (g), waste water from laundry (l), galley waste (w), and dishwashing water (d) is presented in Table 3. A corresponding lattice of rough set approximations can be seen in Fig. 1.

An ordered set for objects: graywater, waste water from laundry, galley waste, and dishwashing water with respect to the \(\not\subseteq\) relation is presented in Fig. 2.

A contraordinal scale for the same objects can be seen in Table 4. A lattice for graywaters appears in Fig. 3.
Table 4: A contraordinal scale.

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<th></th>
<th>graywater</th>
<th>waste water from laundry</th>
<th>galley waste</th>
<th>dishwashing water</th>
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<tr>
<td>dishwashing water</td>
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4 Conclusion

Rough sets approximations turn out to be a very useful tool for classifying greywater. They can be further on used in the process of building intelligent decision support systems.

References:


