A Remark On Taylor Series Method

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Abstract
In this work, we introduced the differential equations of non integer order. Then, we generalized the Taylor series method to special class of equation of order (1, q) by applying the Taylor series method to linear differential equations of order (1, 2), known as semi-differential equations. To established this approach, we compared this result against results obtained by Adomian decomposition method. This study shows that the Taylor series method can be generalized for the differential equations of order (1, q), which, encourages us to study the same approach for other types of linear and non-linear fractional and extra-ordinary differential equations.

Key-words: - Taylor series method, Adomian method, ordinary and fractional differential equations, fractional calculus.

1 Introduction
The most frequently encountered definition of an integral of fractional order is via an integral transform [1-4]. This type of integral is called the Riemann-Liouville integrals. The Riemann-Liouville integrals of \( f(t) \) of order \( \alpha \) are defined as follows

\[
D^{-\alpha} [ f(t) ] = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) \, dx
\]  (1)

In the formula (1.1), \( \alpha \) is a positive real number, \( t > 0 \), \( \Gamma \) is the Gamma function and \( f(t) \) is a function of class \( C^{(\alpha)} \) (the class of functions with continuous \( n^{th} \) derivatives).

To define the Riemann-Liouville fractional derivatives, we assume that \( n \) is the smallest integer exceeding \( \alpha \) such that, \( n - \alpha > 0 \), \( (n = 0, \text{ if } \alpha < 0) \), then the Riemann-Liouville fractional derivatives of the function \( f(t) \) of order \( \alpha \) is defined as

\[
D^{\alpha} [ f(t) ] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left[ (t-x)^{n-\alpha} f(x) \right]
\]  (2)

For \( \alpha > 0 \), \( \beta > 0 \) and a positive integer \( p \), we have the following formulas

\[
D^{\alpha} D^{\beta} [ f(t) ] = D^{\alpha+\beta} [ f(t) ] = D^{\beta} D^{\alpha} [ f(t) ]
\]  (3)

\[
D^{\alpha} f(t) = f(t)
\]  (4)

\[
D^{\alpha} [ D^{\alpha} [ f(t) ] ] = D^{\alpha+\alpha} [ f(t) ]
\]  (5)

\[
D^{\alpha} [ D^{\alpha} [ f(t) ] ] = D^{\alpha+\alpha} [ f(t) ]
\]  (6)

\[
D^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{i=1}^{m} C_i t^{\alpha-i}
\]  (7)

In the formula (7), \( m \) is a positive integer, such that \( m-1 < \alpha \leq m \), \( t > 0 \) and \( C_1, C_2, C_3, \ldots C_m \) are arbitrary constants given by

\[
C_i = D^{\alpha-i} [ f(t) ]_{x=0} = D^{\alpha-i} [ f(0) ]
\]  (8)
As an example, the fractional integrals and derivatives of order $\alpha$ of $f(t) = t^\mu$, $\mu > -1$, $t > 0$ are

$$D^{-\alpha}[t^\mu] = \frac{\Gamma(\mu + 1)}{\Gamma(\alpha + \mu + 1)} t^{\mu + \alpha}$$

$$D^{\alpha}[t^\mu] = \frac{\Gamma(\mu + 1)}{\Gamma(\mu - \alpha + 1)} t^{\mu - \alpha}$$

(9)

2 The non integer order differential equations

In this section, we introduce the differential equations of non integer order [6]. To do that, we first introduce the fractional differential operator of order $(n,q)$ as

$$P \left( D^{\frac{n}{q}} \right) = a_n D^{\frac{n}{q}} + a_{n-1} D^{\frac{n-1}{q}} + \ldots + a_0, \quad a_n \neq 0$$

(10)

Here, the coefficients $a_i$; $(i = 0,1,2,\ldots,n)$ are arbitrary constants, $n$ is a positive integer and we consider $q$ to be a positive integer represents the fractional term with $q > 1$, hence for $q = 1$, (10) is just an ordinary differential operator of order $(n,1)$ or simple of order $n$.

Now, we use (10) to introduce the general form of differential equations of non integer order as

$$\left( a_n D^{\frac{n}{q}} + a_{n-1} D^{\frac{n-1}{q}} + \ldots + a_0 \right) u(t) = 0$$

(11)

Equation (11) contains ordinary and fractional terms, hence we will call this type of equations as extraordinary differential equations of order $(n,q)$. According to (11), we can give the following classification of the non integer order differential equations: (I) If $n = q$, then (11) represents an extraordinary differential equation and the leading term is ordinary. (II) If $n < q$, then (11) represents fractional differential equation and the leading term is fractional. (III) If $n > q$, then (11) represents an extra-ordinary differential equation and the leading term is ordinary or fractional.

Hence, For $n = 1$, equation (11) represents a fractional differential equation of type (II) and of order $(1,q)$. And, for $n = 1 & q = 2$, equation (11) represents a fractional differential equation of order $(1,2)$, which are known as semi-differential Equations. As an example, the equation

$$\frac{d^2 u(t)}{dt^2} + u(t) = 0, \quad t > 0$$

(12)

is a as semi-differential Equation of order $(1,2)$, and of type (II) above, with $n = 1 & q = 2$.

In ref. [7], the Adomian series solution of (12) found in the form

$$u(t) = C_1 \left( \frac{1}{\sqrt{\pi}} t^{\frac{1}{2}} - 1 + \frac{2}{\sqrt{\pi}} t^{\frac{3}{2}} - t + \frac{4}{3\sqrt{\pi}} t^{\frac{5}{2}} - \frac{1}{2} t^3 + \ldots \right)$$

(13)

where $C_1$ is an arbitrary constant. Following ref. [7], the Adomian series solution (12) in term of the error function obtained in the form

$$u(t) = C_1 \left[ e^{-t^{\frac{1}{2}}} e^{\frac{t^{\frac{1}{2}}}{\sqrt{\pi}}}, \quad \text{erf} \left( t^{\frac{1}{2}} \right) - e^{-t^{\frac{1}{2}}} \frac{t^{\frac{1}{2}}}{\sqrt{\pi}} \right]$$

(14)

where the error function is defined as

$$\text{erf} \left( t^{\frac{1}{2}} \right) = e^{-t^{\frac{1}{2}}} \left( \sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(3/2+k)} \right)$$

3 Generalizing Taylor series method

To generalize Taylor series method for differential equations of of order $(1,q)$, let us study the series solution for the semi-differential Equations (12). To find the series solution, let us first rewrite (12) in the operator form

$$D^{\frac{1}{2}} u(t) = -u(t)$$

(15)

then, we operate on both sides of (15) by $D^{\frac{1}{2}}$. This yields

$$D^{\frac{1}{2}} \left( D^{\frac{1}{2}} u(t) \right) = -D^{\frac{1}{2}} u(t)$$

Now the formula (7), for $m = 1$, gives
\[ u(t) - \frac{C_0}{\sqrt{\pi}} t^{-\frac{1}{2}} = -D^2 u(t) \]  

(16)

To find a series solution, let us start with the solution

\[ u(t) = a_1 t^{\frac{1}{2}} + \sum_{i=0}^{\infty} a_i t^i \]  

(17)

By substuting (17) into (16) and making use of (9), we can write

\[ a_1 t^{\frac{1}{2}} + \sum_{i=0}^{\infty} a_i t^i - \frac{C_0}{\sqrt{\pi}} t^{\frac{1}{2}} = -\sqrt{\pi} a_{-1} - \sum_{i=0}^{\infty} a_i \frac{\Gamma \left( \frac{i+1}{2} \right)}{\Gamma \left( \frac{i+3}{2} \right)} t^i, \]

which can be written in the form

\[ \left( \frac{a_1}{\sqrt{\pi}} - \frac{C_0}{\sqrt{\pi}} \right) t^{\frac{1}{2}} + a_0 + \sum_{i=1}^{\infty} a_{-i} t^i = -\sqrt{\pi} a_{-1} - \sum_{i=0}^{\infty} a_i \frac{\Gamma \left( \frac{i+1}{2} \right)}{\Gamma \left( \frac{i+3}{2} \right)} t^i, \]

\[ \sum_{i=0}^{\infty} a_i \frac{\Gamma \left( \frac{i+1}{2} \right)}{\Gamma \left( \frac{i+3}{2} \right)} t^i = -\sqrt{\pi} a_{-1} - \sum_{i=0}^{\infty} a_i \frac{\Gamma \left( \frac{i+1}{2} \right)}{\Gamma \left( \frac{i+3}{2} \right)} t^i, \]

Hence, by using (13), we can assume that

\[ a_0 = -C_i \]  

(18)

This leads to

\[ a_{-1} = \frac{C_i}{\sqrt{\pi}} \]  

(19)

\[ a_{-i} = -\frac{\Gamma \left( \frac{i+1}{2} \right)}{\Gamma \left( \frac{i+3}{2} \right)} a_i, \quad i = 0, 1, 2, \ldots \]

The formula (19) leads to the Adomian solution (14), and hence this gives

\[ a_1 = \frac{2}{\sqrt{\pi}} C_i, \quad a_2 = -C_i, \]

\[ a_3 = \frac{4}{3 \sqrt{\pi}} C_i, \quad a_4 = -\frac{1}{2!} C_i \]

\[ a_5 = \frac{4}{3 \sqrt{\pi}} C_i, \quad a_6 = -\frac{C_i}{3!} \]

(20)

This leads to the series solution (13) and hence (14).
In general, and for a fractional differential equation of the form
\[
\frac{d^q u(t)}{dt^q} + u(t) = 0, \quad t > 0
\]  
which is a fractional differential equation of type (II), and order \((1, q)\), we can start with the series solution
\[
u(t) = \sum_{i=-q}^{\infty} a_i t^i
\]  
Hence, by operating on both sides of (25) by \(D^{-\frac{1}{q}}\) and making use of the formula (7), for \(m=1\), we can find
\[
a_{-q} = \frac{C_i}{\Gamma(1/q)},
\]  
\[
a_{-1} = \frac{\Gamma\left(\frac{1}{q} + \frac{1}{2} + i\right)}{\Gamma\left(\frac{1}{q} + \frac{1}{2}\right)} a_i,
\]  
i = -q, (-q+1),..., -2, -1, 0, 1, 2, ...

This leads to the series solution in the form (26) with coefficients given by (27). Furthermore, for \(q = 1\), (25) reduces to the ordinary differential equation \(u'(t) + u(t) = 0\), and the formula (27) reads
\[
a_{-1} = C_i
\]  
\[
a_{-1} = \frac{\Gamma(i + 2)}{\Gamma(i + 3)} a_i = -\frac{1}{i+3} a_i, \quad i = -1, 0, 1, 2, ...
\]  
This gives
\[
u(t) = C_i \left(1 - \frac{1}{2} t + \frac{1}{6} t^2 - \frac{1}{24} t^3 + \ldots\right)
\]  
\[
= C_i \sum_{i=0}^{\infty} \frac{(-t)^i}{(i+1)!} = C_i e^{-\frac{t^2}{2}}
\]

4 Conclusion
In this work, we introduced the differential equations of non integer order. Then, we generalized the Taylor series method to special class of equation of order \((1, q)\). This study shows, that Taylor series method can be applied to differential equations of order \((1, q)\) and this encourages us to apply the same approach for other types of linear and non-linear fractional and extra-ordinary differential equations.

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