Abstract: The spectral method that we use in our paper is closely related with the finite element method (FEM). The main difference between them is that FEM approximates the solution as a linear combination of piecewise functions, which are nonzero on small sub-domains, while a spectral method approximates the solutions as linear combination of continuous functions that are generally nonzero over the domain of solution. Therefore, the finite element method is a local approach, while the spectral method is a global approach of the phenomenon. The approximate solution of our boundary problem for an integral-differential equation is a trigonometric series with variable coefficients. The comparative study of this solution with the exact solution for a numerical example leads to the conclusions regarding the values of errors.

Key words: Stationary transport equation, integral-differential equation, spectral method, finite difference method.

1 Introduction
This paper deals with a typical problem of the mathematical-physics: the solving of neutral particle transport equation. In a reactor, the neutrons are yielded at the fission of the nucleus and they are named the rapid neutrons with an average speed equals to \(2 \times 10^7\) m/s. The rapid neutrons are subjected to a slowness process, their energy decreasing until these are in an equilibrium state with the others atoms of the environment. When the reactor is in a stationary state, the particles have the tendency to move from a region with a great density to another with a small density and thus on obtain a uniform density. This process is named the diffusion. The main problem in the nuclear reactor theory is to find the neutrons distribution in the reactor, hence its density that is the solution of an integral-differential equation named the neutron transport equation. The resolution of the problems dealing with transport phenomena is the subject of several works: [1]-[14]. The following methods are proposed by these papers: Fourier transform, Laplace transform, the least squares, the finite element, Monte Carlo, truncated series of Chebyshev polynomials.

In this study we approach a spectral method accompanied by finite difference method to solve a boundary value problem for the linearized version of the Boltzmann equation with numerous applications in physics and astrophysics. The comparative analysis of the approximate solution with the exact solution for a numerical example leads to the conclusions regarding the values of errors.
2 Problem formulation

Let us consider the integral-differential equation of transport theory for the stationary case:

\[
\mu \frac{\partial \phi(x, \mu)}{\partial x} + \sigma \phi(x, \mu) = \frac{\sigma}{2} \int_0^1 \phi(x, \mu')d\mu' + f(x, \mu)
\]

\[
\forall (x, \mu) \in D_1 \times D_2 = [0,1] \times [-1,1],
\]

\[
D_2 = D'_1 \cup D''_2 = [-1,0] \cup [0,1]
\]

and the following boundary conditions:

\[
\phi(0, \mu) = 0 \quad \text{if} \quad \mu > 0, \quad \phi(1, \mu) = 0 \quad \text{if} \quad \mu < 0
\]

where

- \( \phi(x, \mu) \) is the density of neutrons, which migrate in a direction that makes an angle \( \alpha \) with the \( x \) axis and \( \mu = \cos \alpha \);
- \( \sigma \) and \( \sigma_s \) are the constants (isotropic region) that verify the inequalities: \( 0 < \sigma_s < \sigma \);
- \( f(x, \mu) \) is a given radioactive source function.

Now, we split the equation (1) in two equations, using the following notations:

\[
\phi^+ = \phi(x, \mu) \quad \text{if} \quad \mu > 0, \quad \phi^- = \phi(x, \mu) \quad \text{if} \quad \mu < 0
\]

Denoting \( \mu'' = -\mu' \), the following integral becomes

\[
\int_0^1 \phi(x, \mu')d\mu' = \int_0^1 \phi(x, -\mu')d\mu'' = \int_0^1 \phi^- d\mu''.
\]

and now the conditions (2) are

\[
\phi^+(0, \mu) = 0, \quad \phi^-(1, \mu) = 0
\]

In view of (3), the equation (1) can be written in the form

\[
\mu \frac{\partial \phi^+}{\partial x} + \sigma \phi^+ = \frac{\sigma}{2} \int_0^1 (\phi^+ + \phi^-)d\mu' + f^+
\]

\[
- \mu \frac{\partial \phi^-}{\partial x} + \sigma \phi^- = \frac{\sigma}{2} \int_0^1 (\phi^+ + \phi^-)d\mu' + f^-
\]

Adding and subtracting the equations (5) and introducing the notations:

\[
u = \frac{1}{2}(\phi^+ + \phi^-), \quad v = \frac{1}{2}(\phi^+ - \phi^-)
\]

\[
g = \frac{1}{2}(f^+ + f^-), \quad r = \frac{1}{2}(f^+ - f^-)
\]

we obtain the following system

\[
\mu \frac{\partial v}{\partial x} + \sigma u = \frac{1}{2} \int_0^1 ud\mu + g \quad (a)
\]

\[
\mu \frac{\partial u}{\partial x} + \sigma v = r \quad (b)
\]

with the boundary conditions:

\[
u + v = 0 \quad \text{for} \quad x = 0,
\]

\[
u - v = 0 \quad \text{for} \quad x = 1.
\]

Determining \( v \) from the second equation of (7) and using the first equation, we rewrite the problem (7) - (8) in the following form

\[
-\mu^2 \frac{\partial^2 u}{\partial x^2} + \sigma u = \frac{1}{2} \int_0^1 ud\mu + g - \frac{\mu}{\sigma} \frac{\partial r}{\partial x}
\]

\[
\begin{cases}
\left( u - \frac{\mu}{\sigma} \frac{\partial u}{\partial x} \right)_{x=0} = -\frac{r(0, \mu)}{\sigma} \\
\left( u + \frac{\mu}{\sigma} \frac{\partial u}{\partial x} \right)_{x=1} = \frac{r(1, \mu)}{\sigma}
\end{cases}
\]

where \( \mu \in [0, 1] \).

Let us now suppose that the functions \( u, g \) and \( r \) belong to the Hilbert space \( L_2(D) \), where \( D = D_1 \times D_2^* \) and the scalar product is defined by the formula

\[
\langle w, t \rangle = \int_0^1 w(x) t(\mu) dx d\mu
\]

Let us now consider the function \( U(x) = u(x, \mu) \) for a fixed \( \mu \in [0, 1] \), where \( U(x) \) is square integrable in \([0, 1]\). In the spectral methods, \( U(x) \) may be approximated by using orthogonal functions, \( \alpha_k(x) \), as

\[
U(x) = \sum_{k=0}^\infty \alpha_k \omega_k(x)
\]
where $\alpha_k$ are coefficients. In practice, it is used for $U$
the approximation

$$U(x) \approx \sum_{k=0}^{n} \alpha_k \omega_k(x)$$  (13)

The coefficients $\alpha_k$ are chosen, such that to minimize
the mean integral square error

$$e = \int_0^1 \left| U(x) - \sum_{k=0}^{n} \alpha_k \omega_k(x) \right|^2 dx$$  (14)

and are given by

$$\alpha_k = \int_0^1 U(x) \omega_k(x) dx$$  (15)

Next, we take

$$\omega_k(x) = \cos 2k\pi x$$  (16)

and the solution of our problem (9) – (10)

$$u_n(x, \mu) = \sum_{k=0}^{n} a_k(\mu) \cos 2k\pi x$$  (17)

On the boundary we have $\sum_{k=0}^{n} a_k(\mu) = 0$ for the odd
function $r$ that is of the form

$$r(x, \mu) = \beta(\mu) - \beta(\mu) \cos 2\pi x + \gamma(\mu) \sin 2\pi x$$  (18)

where $\beta$ and $\gamma$ are known function obtained with help
of the source function.

If the solution (23) is introduced in equation (9), then
using the spectral – Galerkin method, we get

$$\left( \int_0^1 F(x, \mu) dx \right) \cos 2k\pi x - \left( \int_0^1 F(x, \mu) \cos 2k\pi x dx \right) = \sum_{l=1}^{m-1} a_l(\mu_l)$$

$$= \sum_{l=1}^{m-1} a_l(\mu_i) + \frac{1}{2} a_j(\mu_0) + \frac{1}{2} a_j(\mu_m)$$

$$= G_j(\mu_i)$$  (22)

Solving these $n+1$ systems, we can evaluate the solution
values of our problem (9) - (10) in the nodes $\mu_i$

$$u_n(x, \mu_i) = \sum_{k=0}^{n} a_k(\mu_i) \cos 2k\pi x, \ i \in \{0,1,...,m\}$$  (23)

In order to get $v(x, \mu_i), i \in \{0,1,...,m\}$, we use (7b)
and the finite difference method to calculate the partial
derivative. For this purpose, we consider a partition of the interval $D_i$ with step $h$: $0 = x_0 < x_1 < ... < x_p = 1$.

The values of $v$ in nodes $(x_s, \mu_l), s \in \{1,2,...,p-1\}, \ l \in \{0,1,...,m\}$ are

$$v(x_s, \mu_l) = \frac{r(x_s, \mu_l)}{\sigma} - \frac{\mu}{\sigma} \frac{u(x_{s+1}, \mu_l) - u(x_{s-1}, \mu_l)}{2h}$$  (24)

Finally, the values of function $\varphi^+$ are determined in
the nodes $(x_s, \mu_l)$ by the formula

$$\varphi^+(x_s, \mu_l) = u(x_s, \mu_l) + v(x_s, \mu_l), \ s \in \{1,2,...,p-1\}, \ l \in \{0,1,2,...,m\}$$  (25)
3 Numerical results

Let us consider the boundary value problem

\[ \mu \frac{\partial \phi(x, \mu)}{\partial x} + 3\phi(x, \mu) = \frac{1}{\mu} \phi(x, \mu') d\mu' + f(x, \mu) \]  

(26)

\[ \phi(0, \mu) = 0 \quad \text{if} \quad \mu > 0 \]

\[ \phi(1, \mu) = 0 \quad \text{if} \quad \mu < 0 \]  

(27)

where

\[ f(x, \mu) = (2\pi\mu^2 - 1.5\mu - 1) \cos 2\pi x + (\pi\mu^2 + 2\pi\mu + 3\mu) \sin 2\pi x + (1.5\mu + 1) \]  

(28)

It follows from the method presented in section 2 that the problem (26) – (27) can be rewritten as

\[ -\mu^2 \frac{\partial^2 u}{\partial x^2} + 3u = 2 \int_0^1 u d\mu' + g - \frac{\mu^2}{3} \frac{\partial r}{\partial x} \]  

(29)

\[ \left( u - \frac{\mu^2}{3} \frac{\partial u}{\partial x} \right)_{x=0} = -\frac{r(0, \mu)}{3} \]

\[ \left( u + \frac{\mu^2}{3} \frac{\partial u}{\partial x} \right)_{x=1} = \frac{r(1, \mu)}{3} \]  

(30)

where

\[ g(x, \mu) = (2\pi\mu^2 - 1) \cos 2\pi x + \pi\mu^2 \sin 2\pi x + 1 \]

\[ r(x, \mu) = -1.5\mu \cos 2\pi x + (2\pi + 3\mu) \sin 2\pi x + 1.5\mu \]

Let us consider \( n = 4 \) in (17) and \( m = 4 \) in (22). In matrix form, for each \( j \in \{0, 1, \ldots, n\} \), we obtain a system

\[ A_j \cdot a_j = G_j \]  

(31)

The matrices \( G_j \) will be calculated using function

\[ F(x, \mu) = (-1 - 4\pi^2 \mu^2 / 3) \cos 2\pi x + 1 \]  

(32)

Solving these \( n + 1 \) systems, we find the values of each \( a_j \) in the nodes \( \mu_l, l \in \{0, 1, \ldots, 4\} \) and using (17) we obtain the approximate solution \( u_n \) in the nodes \((x_s, \mu_l)\)

\[ u_n(x_s, \mu_l) = \sum_{k=0}^{4} a_k(\mu_l) \cos 2k\pi x_s \]  

(33)

Then, with formulas (24) and (25) we calculate \( \varphi(x_s, \mu_l), s \in \{0, 1, \ldots, 8\}, l \in \{0, 1, \ldots, 4\} \). These approximate values of \( \varphi \) are compared with their exact values:

\[ \varphi ex(x_s, \mu) = (\mu + 2) \sin^2 x_s \pi + \mu \sin 2x_s \pi \]  

(34)

The surfaces that correspond to \( u_n(x_m, \mu_l) \) and exact even function \( u(x_m, \mu_l) \) (denoted by \( u \) and \( uex \), \( m \in \{0, 1, \ldots, 8\} \), \( l \in \{0, 1, \ldots, 4\} \) are shown in the Fig. 1. These two even functions practically coincide.

Surfaces of odd functions: \( v \) – approximate function (green color) and \( vex \) – corresponding exact function (yellow color) are shown in Fig. 2. The form of density surfaces: \( \varphi \) (fi – green color) and \( \varphi ex \) (fiex-brown color) can be seen in Fig. 3.
The Fig. 4 shows the variations of numerical solution $\varphi$: $fi$ - red and green color and exact solution $\varphi_{ex}$: $fiex$ - blue and brown color for $\mu \in \{0.25; 0.5\}$. The same solutions are shown in Fig. 5 for $\mu \in \{0.75; 1.\}$. We observe that the values of $\varphi$ increase with increasing of $\mu$.

To verify the accuracy of results obtained with our algorithm we calculate the errors:

$$erfi(x_m, \mu_l) = \varphi_{ex}(x_m, \mu_l) - \varphi(x_m, \mu_l)$$  \hspace{1cm} (35)  

$m \in \{0,1,...,8\}, l \in \{0,1,...,4\}$

The absolute value of error increases with the increase of $\mu$. We denoted in Fig.6: $erfi_{m,1}$ – errors for $\mu = 0.25$; $erfi_{m,2}$ – errors for $\mu = 0.5$ and $erfi_{m,3}$ – errors for $\mu = 0.75$. 

Fig. 3

Fig. 4

Fig. 5

Fig. 6
The values of numerical solution, exact solution and the errors ($er$) that correspond to the above figures 4, 5 and 6 are presented in Table 1 and Table 2.

Table 1. The computational results

<table>
<thead>
<tr>
<th>$x_m$</th>
<th>$\varphi_{\mu=1/4}$</th>
<th>$\varphi_{ex\mu=1/4}$</th>
<th>$er_{\mu=1/4}$</th>
<th>$\varphi_{\mu=1/2}$</th>
<th>$\varphi_{ex\mu=1/2}$</th>
<th>$er_{\mu=1/2}$</th>
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<td>$x_3$</td>
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<td>-0.01</td>
<td>2.6</td>
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<td>2.5</td>
<td>2.5</td>
<td>0</td>
</tr>
<tr>
<td>$x_5$</td>
<td>1.7</td>
<td>1.74</td>
<td>0.04</td>
<td>1.71</td>
<td>1.77</td>
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<td>$x_6$</td>
<td>0.83</td>
<td>0.87</td>
<td>0.04</td>
<td>0.68</td>
<td>0.75</td>
<td>0.07</td>
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<tr>
<td>$x_7$</td>
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<td>0.01</td>
<td>0.02</td>
<td>0.01</td>
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<td>$x_8$</td>
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Table 2. The computational results

<table>
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<th>$\varphi_{ex\mu=3/4}$</th>
<th>$er_{\mu=3/4}$</th>
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<td>1.71</td>
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<tr>
<td>$x_6$</td>
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5 Conclusions

In this paper a numerical algorithm based on a spectral approach and the finite difference schemes was presented for a source function of the form

$$f(x, \mu) = \alpha(\mu) \cos 2\pi x + \beta(\mu) \sin 2\pi x + \gamma(\mu)$$

The results presented in Table 1 and Table 2 show that even using only five terms in (23) and five nodes in trapezoidal approximation, the solutions obtained with our algorithm have acceptable values. This algorithm can be easily extended to a boundary value problem for a two-dimensional transport equation.

References