Identification of Parameters of a System with Fractional Damping

Liviu Bereteu, Gheorghe E. Drăgănescu, Dan V. Stănescu
Department of Mechanics and Material Strength
Polytechnica University of Timişoara
Bd M. Viteazu No 1, RO 300222
ROMANIA
liviu.bereteu@mec.upt.ro http://www.mec.upt.ro

Abstract: - In this paper is introduced a useful method for identification of parameters of a vibrating system with fractional damping. The method is based on the wavelet multiresolution analysis and can be used to identify the systems with one or many degree of freedom.

Key-Words: - parameter identification, fractional derivative, fractional damping, wavelet multiresolution analysis, mother wavelet, vibrations

1 Introduction
Last years, in mechanics, it was developed a series of mathematical models by coming out the fractional derivatives [1, 2]. These models are especially used in rheological description of plastic materials [3], and in dielectric spectra, nanosystems, etc.

In this moment, the number of papers devoted to the identification of parameters of these models is limited [4-7].

In this paper we will introduce a new method for parameter identification of the fractional degree systems, including the fractional derivatives.

2 Problem Formulation
Here is the outline of an identification method for a system described by a differential equation which represents a generalization of the following equation:

\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) + N(x, x) = F(t) \]  

having a following fractional form:

\[ m\ddot{x}(t) + c\dot{x}(t) + r_{a}D^{\alpha}_{t}x(t) + kx(t) + \]

\[ N(x(t), \frac{dx(t)}{dt}) = F(t) \]  

\[ \alpha \] represents a non-integer positive number and \( r \) is a real parameter.

It was found that exist polymers inside which occurs a damping phenomenon described by a fractional term like the one in the form (2). It is needful to say that the equation is easy to extend to cases involving more terms with fractional order derivatives.

In a strict manner, the equation (2) may be written as:

\[ m\ddot{x}(t) + c\dot{x}(t) + r_{a}D^{\alpha}_{t}x(t) + kx(t) + \]

\[ N(x(t), \frac{dx(t)}{dt}) = F(t) \]  

where fractional derivation operator \( r_{a}D^{\alpha}_{t} \) is relied to the real parameter \( a \).

3 Wavelet multiresolution analysis
The multiresolution wavelet analysis was introduced by Mallat and Meyer. This method can be used in vibration analysis [8,9,10].

It was found that a signal \( x(t) \), representing a square integrable function \( x \in L_{2}(R) \), can be expanded in series in terms of a basis \( e_{nl} = \Psi(2^{n}t-l) \in L_{2}(R) \)

builted from a mother wavelet function \( \Psi(t) \) in the form [8]:

\[ x(t) = \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} <x(t) | \Psi(2^{n}t-l) > \Psi(2^{n}t-l) \]

\[ = \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \alpha_{nl} \Psi(2^{n}t-l), \]

where \( <\Psi|\Phi> \) represents the inner product

\[ \Phi | \Psi > = \int_{-\infty}^{\infty} \Phi^{*}(t)\Psi(t)dt, \]

with *- complex conjugate.

The coefficients \( \alpha_{nl} \) of the series can be calculated as:

\[ \alpha_{nl} = <x(t) | \Psi(2^{n}t-l) > = \int_{-\infty}^{\infty} x^{*}(t)\Psi(2^{n}t-l)dt. \]
The series (3) can be written in a form more useful for calculational purposes using wavelet multiresolution analysis as:

\[ x(t) = \beta_0 \Phi(t) + \sum_{l=0}^{n} \beta_{2^n+l} \Psi(2^n t - l), \quad (5) \]

where \( \Phi(t) \) represents a scale function and between the coefficients can be written the relation: \( \beta_{2^n+l} = \alpha_{nl} \).

The equation (5) can be re-written as a series of matrix [8]:

\[ x(t) = \beta_0 \Phi(t) + \beta_1 \Psi(t) + \ldots + \beta_{2^n} \Psi(2^n t - l) + \ldots, \quad (6) \]

or in the final form:

\[ x(t) = \beta_0 \Phi(t) + \beta_1 \Psi_1(t) + \beta_2 \Psi_2(t) + \ldots + \beta_n \Psi_n(t) + \ldots, \quad (7) \]

The coefficients \( \beta_i \) can be calculated using the relations

\[ \beta_0 = \int_{-\infty}^{\infty} x(t) \Phi(t) dt, \]
\[ \beta_{2^n+l} = 2^n \int_{-\infty}^{\infty} x(t) \Psi(2^n t - l) dt, \quad (8) \]

where \( 2^n + l \neq 0 \).

4 Problem Solution

It is necessary to highlight that the fractional derivation operator \( _a D_t^\alpha \) is a linear operator.

We can write the fractional derivative, in the form of a finite difference equation, on a finite time range \( \Delta t \). With this aim we will use the definition Grünwald-Letnikow for a fractional order derivative:

\[ _a D_t^\alpha = \lim_{\Delta t \to 0} \frac{1}{\Delta t^\alpha} _a \Delta_t^\alpha x(t) \quad (9) \]

where the finite difference operator \( _a \Delta_t^\alpha \) is written:

\[ _a \Delta_t^\alpha x(t) = \sum_{j=0}^{v} (-1)^j \binom{\alpha}{j} x(t - j\Delta t) \quad (10) \]

where \( v \) represents the integer part of:

\[ v = \left\lfloor \frac{t - a}{\Delta t} \right\rfloor, \]

and \( \binom{\alpha}{j} \) represents the generalized binomial coefficient, which can be written with the Euler's gamma function:

\[ \binom{\alpha}{j} = \frac{\Gamma(\alpha + 1)}{j!\Gamma(\alpha - j + 1)}. \]

The parameter \( \alpha \) denotes the origin of the time on which the fractional derivative is defined. So, there are systems without memory or with short memory, on which the history of the process is not so important like in the case of hysteretic processes. In the case of short memory systems, having a memory limited on \( T \) time span, the fractional derivative can be defined on a limited range, which is moving meantime with the displacement of time \( t \). It results that the fractional derivative can be written, for this case, in the following approximate form:

\[ _{t-\tau} D_t^\alpha x(t) = \frac{1}{\Delta t^\alpha} \sum_{j=0}^{v} (-1)^j \binom{\alpha}{j} x(t - j\Delta t) \quad (11) \]

where \( v = [T/\Delta t] \). \( T \) can be taken so that the fractional derivative will be computed for a reduced set of points, for example \( v = 2, 3, 4, 5 \).

Including (11) in (3) it results:

\[ m \frac{d^2x(t)}{dt^2} + c \frac{dx(t)}{dt} + \sum_{j=1}^{v} \rho_j x(t - j\Delta t) + k x(t) + N(x(t), \frac{dx(t)}{dt}) = F(t) \quad (12) \]

where the coefficients \( \rho_j \) are:

\[ \rho_j = \frac{r}{\Delta t^\alpha} (-1)^j \binom{\alpha}{j} \]

In the next table there are shown the first values of the \( \rho_j \) coefficients:

<table>
<thead>
<tr>
<th>( j )</th>
<th>( \rho_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{r}{\Delta t^\alpha} )</td>
</tr>
<tr>
<td>1</td>
<td>( -\frac{r}{\Delta t^\alpha} \alpha )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{r}{2\Delta t^\alpha} \alpha(\alpha - 1) )</td>
</tr>
<tr>
<td>3</td>
<td>( -\frac{r}{6\Delta t^\alpha} \alpha(\alpha - 1)(\alpha - 2) )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{r}{24\Delta t^\alpha} \alpha(\alpha - 1)(\alpha - 2)(\alpha - 3) )</td>
</tr>
<tr>
<td>5</td>
<td>( -\frac{r}{120\Delta t^\alpha} \alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4) )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{r}{720\Delta t^\alpha} \alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)(\alpha - 5) )</td>
</tr>
</tbody>
</table>
The non-linear function (1) can be developed in exponent series by the position \( x \) and instant velocity \( \dot{x} \). That developing can be made for example in Taylor series around the origin, resulting:

\[
m\ddot{x}(t) + c\dot{x}(t) + a \mathcal{D}_x^0 x(t) + kx(t) + \sum_{n=2}^{\infty} \alpha_n x^n(t) + \sum_{m=2}^{\infty} \beta_m \dot{x}^m(t) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \gamma_{n+m} x^n(t) \dot{x}^m(t) = F(t)
\]  

(13)

For the identification of the parameters of the equation (13), the non-linear term can be developed:

\[
m\ddot{x}(t) + c\dot{x}(t) + \sum_{j=1}^{\infty} \rho_j x(t - j\Delta t) + kx(t) + \sum_{n=2}^{\infty} \alpha_n x^n(t) + \sum_{m=2}^{\infty} \beta_m \dot{x}^m(t) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \gamma_{n+m} x^n(t) \dot{x}^m(t) = F(t)
\]  

(14)

Also the terms of this equation can be developed in wavelet function series. Furthermore, the delayed function \( x(t - j\Delta t) \) (where \( j = 1, 2, ..., \nu \)) can be developed as:

\[
x(t - j\Delta t) = \sum_{n=1}^{\infty} \{j, n\} \Psi_n = \sum_{n=1}^{\infty} \int_{t-j\Delta t} x(t) \Psi_n(t) \, dt
\]  

(15)

The derivatives of \( x(t) \) and the powers from (15) can be expressed in terms of series according (8) as:

\[
\begin{align*}
\dot{x} &= \sum_{n=1}^{\infty} < \dot{x} >_n \Psi_n = \sum_{n=1}^{\infty} \dot{x}_n \Psi_n, \\
\ddot{x} &= \sum_{n=1}^{\infty} < \ddot{x} >_n \Psi_n = \sum_{n=1}^{\infty} \ddot{x}_n \Psi_n, \\
x &= \sum_{n=1}^{\infty} < x >_n \Psi_n = \sum_{n=1}^{\infty} x_n \Psi_n, \\
x^2 &= \sum_{n=1}^{\infty} < x^2 >_n \Psi_n = \sum_{n=1}^{\infty} x_n^2 \Psi_n, \\
x^k &= \sum_{n=1}^{\infty} < x^k >_n \Psi_n = \sum_{n=1}^{\infty} x_n^k \Psi_n, \\
\dot{x}^2 &= \sum_{n=1}^{\infty} < \dot{x}^2 >_n \Psi_n = \sum_{n=1}^{\infty} \dot{x}_n^2 \Psi_n, \\
\ddot{x}^m &= \sum_{n=1}^{\infty} < \ddot{x}^m >_n \Psi_n = \sum_{n=1}^{\infty} \ddot{x}_n^m \Psi_n,
\end{align*}
\]

for \( k, m = 1, 2, 3, 4 \), Replacing in (14), we can extract a relation between the coefficients for the same \( \Psi_n \).

It results the matrix equation:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_\nu \\
\end{pmatrix} = \begin{pmatrix}
m \\ c \\ \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ k \\
\end{pmatrix} \begin{pmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
F_5 \\
F_6 \\
\end{pmatrix}
\]

(17)

and this can be written on the formal form:

\[
[|X|][q] = [F]
\]

(18)

where \([|X|]\) is a squared infinite matrix, \([q]\) is a column vector representing the model's parameters and \([F]\) is a column vector representing the development coefficients of the perturbing force.

The system (18), being overdetermined, the solutions \([q]\) can be determined with the rule of the least squares, resulting:

\[
[q] = \left(|X|^{-1} |X| \right)^{-1} [F]
\]

(19)

where \([.|.|^{-1}]\) denotes the reversed matrix and \([.|.|]\) the transposed matrix.

From the experimental point of view, for a given system the perturbation force \( F(t) \) and the response \( x(t) \), can be practically measured and introduced in a sampled form in the computer, via an analog to digital device. In
a numeric way, there can be calculated the derivatives \( \dot{x}, \ddot{x} \) of \( x(t) \). Numerically, we can calculate the coefficients of the series (15), (16), for example in terms of Haar, Daubechies, Coiflet or Simlet basic wavelet. It also results the equation (17) or (18). Finally, we obtain the parameters of the system using (19).

It is important to underline that we can establish the parameters of the system and the order of the fractional derivative.

5 Conclusion

It is introduced a parameter identification method for a fractional nonlinear oscillator, described by (13) or (14).

Practically, for a given system subjected to a perturbation force \( F(t) \) it can be measured the response \( x(t) \), as signals in discrete form. These signals can be expanded in terms of wavelet series (15), (16), from which it results the matrix equation (17).

Finally we obtain the parameters of system using (19) and the order of the fractional derivative.

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References:


