Uncertainty propagation using polynomial chaos and centre manifold theories

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Abstract: - This paper proposes a new methodology for uncertainty quantification in the field of nonlinear dynamic system analysis. It consists in combining both the centre manifold theory and the polynomial chaos approach. The first one is known to be a powerful tool for model reduction of nonlinear dynamic systems in Hopf bifurcation point neighbourhood while the polynomial chaos approach is an efficient tool for uncertainty propagation. Therefore, to couple the two methods can help to overcome computational difficulties due to both the complexity of nonlinear dynamic systems and the cost of the uncertainty propagation with the prohibitive Monte Carlo method. The feasibility and efficiency of the proposed methodology is investigated. So, a two degree of freedom model describing a drum brake system subject to uncertain initial conditions is considered.

Key-Words: - Nonlinear Dynamic Systems, Model reduction, Centre Manifold, Uncertainty Quantification, Polynomial Chaos, Non-intrusive methods, Monte Carlo method.

1 Introduction

Uncertainty propagation plays a major role in the robust analysis and design of dynamic systems. It consists of several techniques designed meant to quantify the influence of different kinds of uncertainties (design parameters, initial conditions, inputs) on the process states and outputs or to take uncertainties into account in the design of processes or control systems. There are several theories about this topic in the literature, such as probabilistic approaches [22-23], [4], fuzzy logic [24] and the interval theory [25]. Among the probabilistic approaches, the Monte Carlo method is the most useful. The latter can give the entire probability density function of any system variable, but it is often too expensive since a large number of samples are required for reasonable accuracy. A more efficient probabilistic tool has been presented in the literature. It consists of the polynomial chaos [9]. This theory was pioneered by Ghanem and Spanos who used expansion in Wiener-Hermite polynomials to model stochastic processes with Gaussian random variables [4]. The convergence of such expansion in a mean square sense has been shown [2] and generalized to various continuous and discrete distributions using orthogonal polynomials following the so called Askey-scheme [7]. The capability of polynomial chaos expansions has been tested on numerous applications, such as solving ordinary and partial differential equations [6-7] and [12], observer and controller design problems [10], [15] and [3]. Polynomial chaos gives a mathematical framework to separate the stochastic components of a system response from deterministic ones. The stochastic Galerkin method [4], [1], collocation and regression methods [14] are used to compute the deterministic components called stochastic modes in an intrusive and a non intrusive manner while random components are concentrated in the polynomial basis used. Non intrusive procedures are shown to be more efficient for stochastic dynamic systems since they need only simulations corresponding to particular samples of the random parameters and they need no modifications on the system model contrary to intrusive methods in which Galerkin’s techniques are used to generate, from the stochastic system model, a set of deterministic coupled equations which are difficult to implement especially for non-linear systems. Generally speaking, the analysis of uncertain dynamic systems using both the Monte Carlo and non-intrusive polynomial chaos methods requires deterministic simulations of the system studied. These simulations are based on numerical integration procedures which are too expensive and difficult tasks with non-linear systems as these have hard non-linearities and numerous degrees of freedom. The model reduction theory proposes a number of solutions consisting of methodologies which approximate complex models with simpler ones while keeping the same dynamic behaviours and the most important properties of the original models. Balanced truncation [16-17], proper orthogonal decomposition (POD) [18] and singular perturbation [19] based methods are well known examples.
The centre manifold is another method presented in literature as an efficient tool which helps to simplify a complex dynamic system in a Hopf bifurcation point neighbourhood \[20\], \[5\], \[11\]. This approach is based on the idea that all dynamic characteristics near the equilibrium point are governed by the dynamics on the centre manifold when some eigenvalues have zero real parts and all the other eigenvalues have negative real parts. The originality of this paper lies in the new methodology proposed to perform a simpler robust analysis of uncertain dynamic systems. The main principle of this methodology is to combine centre manifold theory with the polynomial chaos approach. The first one provides a powerful tool to obtain a reduced model in the Hopf bifurcation neighbourhood, then the main idea is to propagate uncertainty on this reduced model instead of the original one, using a polynomial chaos based approach which is less expensive than the prohibitive Monte Carlo procedure. The objective of this paper is to illustrate the feasibility of the proposed method. So, a two degree of freedom model describing a drum brake system is considered, with uncertainty in the initial conditions. First, the essential principles of both the centre manifold method and the polynomial chaos are summed up in Section 2. The combination of the two methods is tested on a simple self-excited mechanism. All the results are presented in Section 3. Finally, there are some observations and conclusions about perspectives and further research work.

2 Theoretical methods

Let

\[ \dot{x} = f(x, \mu) \]  

be a nonlinear dynamic system for which \( x(t) \in \mathbb{R}^d \) is defined as the state vector and \( \mu \in \mathbb{R} \) a control parameter. The vector field \( f \) is assumed to be smooth, i.e., \( f \in C^\infty \) and the origin is, without loss of generality, the equilibrium point of (1) i.e. \( f(0, \mu) = 0 \). Additionally, here, only polynomial nonlinearities are considered. This is not restrictive since any smooth nonlinear function can be approximated by a polynomial function using the multi-dimensional Taylor series expansion.

2.1 Centre manifold

The centre manifold method uses the basic idea that the essential non-linear dynamic system characteristic in the neighbourhood of an equilibrium point is governed by the centre manifold associated with the part of the original system characterized by the eigenvalues with zero real-parts at the Hopf bifurcation point \[11\]. The system (1) can be expressed by:

\[ \dot{x} = A(\mu)x + F(x, \mu) \]  

where \( F \) is the vector of polynomial functions in \( x \) and \( \mu \). The system (2) can be put in a canonical form (3) at the Hopf bifurcation point \( \mu_0 \) by means of a linear basis transformation

\[ x = T \begin{bmatrix} y \\ z \end{bmatrix} \] 

where \( y \in \mathbb{R}^{n_c}, z \in \mathbb{R}^{n_s} \) such that \( n_c + n_s = n \),

\[ A_c = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_{n_c-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n_c} \end{bmatrix}, \quad A_s = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad F_c(y, z, \mu_0), F_s(y, z, \mu_0) \] 

being vectors of non-linear terms with higher orders such that:

\[ F_c(0, 0, \mu_0) = 0, \quad D_{y,z}F_c(0, 0, \mu_0) = 0 \]  

and,

\[ F_s(0, 0, \mu_0) = 0, \quad D_{y,z}F_s(0, 0, \mu_0) = 0 \]  

denoting the jacobian operator.

In the neighbourhood of the Hopf bifurcation point, the system (3) may be defined by the following augmented dynamics:

\[ \begin{aligned} \dot{y} &= A_c(\tilde{\mu})y + F_c(y, z, \tilde{\mu}) \\ \dot{z} &= A_s(\tilde{\mu})z + F_s(y, z, \tilde{\mu}) \\ \dot{\tilde{\mu}} &= 0 \end{aligned} \]  

where \( \tilde{\mu} = \mu_0(1 + \varepsilon) \) with \( \varepsilon << 1 \).

With the centre manifold theory \[20\],[5] and \[11\], it is demonstrated that for \( \|y\| \) and \( \|\tilde{z}\| \) there is a local centre manifold which helps to express the stable variables \( z \) as
a function of the centre variables \((y, \mu)\) such that:
\[
z = h(y, \mu) \quad \text{where} \quad h \text{ is a function verifying} \quad h(0, 0) = 0 \quad \text{and} \quad D_{(y, \mu)} h(0, 0) = 0.
\]
Consequently, the original system (4) can be reduced to the following reduced model:
\[
\begin{align*}
\dot{y} & = A(y, \mu) y + F_c(y, h(y, \mu), \mu) \\
\dot{\mu} & = 0
\end{align*}
\]
(5)
If \(h\) is chosen to be a polynomial function with a fixed order then it will be a solution of the following equation:
\[
\dot{z} = D_{(y, \mu)} \begin{bmatrix} \dot{y} \\ \dot{\mu} \end{bmatrix}
\]
which is equivalent to:
\[
(D_{(y, \mu)} h) (A(y, \mu) y + F_c(y, h(y, \mu), \mu)) = A(y, \mu) h(y, \mu) + F_c(y, h(y, \mu), \mu)
\]
(6)

### 2.2 Polynomial chaos

From the Wiener theory and the generalized Cameron-Martin theorem, any second order process \(x\) can be expanded in a convergent (in the mean square sense) polynomial function series as:
\[
x = \sum_{i=0}^{\infty} \bar{x}_i \phi_i (\xi)
\]
(7)
\(\xi\) is a vector of \(d\) independent random variables with a known joint density function \(W(\xi)\), \(\bar{x}_i\) are the stochastic modes of the random process \(x\) and \(\phi_j\) are orthogonal polynomial functions satisfying the following relations:
\[
\langle \phi_i, \phi_j \rangle = \int \phi_i (\xi) \phi_j (\xi) dP(\omega) = \begin{cases} 0 & \text{if } i \neq j \\ \phi_i^2 & \text{if } i = j \end{cases}
\]
(8)
\(\langle \cdot \rangle\) denoting the inner product operator

In practice, the polynomial chaos expansion is truncated to a finite order \(P\) which has been demonstrated to be dependent on the polynomial chaos order \(r\) and the stochastic dimension \(d\) which denotes in practice the number of uncertain parameters [4].
\[
X(\omega) \approx \sum_{i=0}^{P} \bar{x}_i \phi_i (\xi(\omega))
\]
(8)
The problem can then be summed up by computing the stochastic modes using intrusive and non-intrusive approaches. Only non-intrusive methods will be considered in this paper and be presented in the following subsection. So, consider a uniform distribution for the initial condition \(x_i(0)\) of the system (1). As consequence, all the state variables also become uncertain. According to the Askey scheme [7], Legendre polynomials are more suitable for uniform uncertainties. So, state variables can be expanded in terms of Legendre polynomials in the standard stochastic variable \(\xi\) which follows a uniform distribution within the orthogonality interval \([-1,1]\) of Legendre polynomials.
\[
x_i (t, \xi) = \sum_{j=0}^{P} \bar{x}_j (t) L_j (\xi), i = 1, ..., n
\]
(9)
The main problem is now to compute the stochastic modes \(\bar{x}_i (t)\). Two methods are presented below.

#### 2.2.1 Non-intrusive spectral projection NISP

The NISP method uses the inner product of \(x_i (t, \xi)\) with \(L_j (\xi)\) and uses the orthogonality of the polynomial basis to compute the stochastic modes as follows:
\[
\bar{x}_{i,j} (t) = \frac{1}{\langle L_j (\xi) \rangle} \langle x_i (t, \xi) L_j (\xi) W(\xi) d\xi \rangle
\]
(10)
Monte Carlo and collocation techniques are used to compute the integral in the numerator of (10), [14]. In this paper, the Gauss collocation method is used.

#### 2.2.2 Regression method

The regression method consists in calculating \(\bar{x}_{i,j}\) by minimizing a least square criterion given by:
\[
\varepsilon = \sum_{i=1}^{q} \left[ x_i (t, \xi^{(i)}) - \sum_{j=0}^{P} \bar{x}_{i,j} (t) \phi_j (\xi^{(i)}) \right]^2
\]
(10)
where \(q\) is a given number of deterministic simulations of the system (1) such that \(P + 1 < q\). \(\{\xi^{(i)}\}\) is a set of simulation points chosen here as the zeros of a Legendre polynomial with a suitable order.

#### 2.2.3 Statistics estimation

Once the stochastic modes are obtained, they are used to determine the statistics of the system variables. The first and second order moments are given respectively by:
\[
\begin{align*}
\bar{x}_{i}^{\text{mean}} (t) & = \bar{x}_{i,0} (t) \\
\sigma_{i}^2 (t) & = \sum_{j=0}^{P} \left( \bar{x}_{i,j} (t) \right)^2 \langle L_j^2 (\xi) \rangle - \left( \bar{x}_{i,0} (t) \right)^2
\end{align*}
\]
(11)

### 3 Application and results

The objective of this paper is to show that the combination of the centre manifold and polynomial
Chaos methods can be an efficient tool to simplify uncertainty propagation problems. Therefore, the estimation of the short term statistics (mean value and standard deviation) of a nonlinear dynamic system subject to uncertain initial conditions has been considered. The estimation of the long term statistics is a more complicated problem which is not dealt with in this paper. A two degree of freedom model describing a drum brake system developing flutter instability (Fig.1) is used. After a short description of the Hultén model, the centre manifold method is used to obtain a reduced order model. The polynomial chaos method will help to estimate the statistics of the original system.

### 3.1 System description

The Hultén model [13], [21] represented in Fig.1 is a self excited mechanism composed of a mass \( m \) held against a moving band; the contact between the mass and the band is modelled by two plates supported by two different springs. For simplicity’s sake, it is assumed that the mass and band surfaces are always in contact. By considering that \( X_1 = x_1, \dot{X}_1 = x_2, X_2 = x_3 \) and \( \ddot{X}_2 = x_4 \), a state space representation for the mechanical system is obtained as:

\[
\dot{x}(t) = A(\mu)x(t) + f_{NL}(x(t), \mu)
\]  
(12)

where 
\[
x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T
\]

\[
A(\mu) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -\eta_1\omega_1 & \mu_1\omega_1^2 & 0 \\ 0 & 0 & 0 & 1 \\ -\mu_2\omega_2^2 & 0 & -\omega_2^2 & -\eta_2\omega_2 \end{bmatrix}
\]

\[
f_{NL}(x(t), \mu) = \begin{bmatrix} 0 \\ -\phi_1^{NL} x_1(t) + \mu_2^{NL} x_3(t) \\ 0 \\ -\mu_2^{NL} x_1(t) - \phi_2^{NL} x_3(t) \end{bmatrix}
\]

with \( \eta_i = c_i / \sqrt{mk_i} \) as the relative damping coefficients, \( \omega_i = \sqrt{k_i / m} \) denoting the natural pulsations and \( \phi_i^{NL} = k_i^{NL} / m \) with \( i = 1, 2 \) and \( \mu \) as the friction coefficient.

### 3.2 Application of the centre manifold method

As mentioned in Section 2.1, the centre manifold method gives a powerful formalism which helps to reduce a nonlinear dynamic system near equilibrium in the neighbourhood of a Hopf bifurcation point. It can be noted that the origin is the equilibrium for the system (12). Moreover, a classical parametric stability study of stability shows that \( \mu_0 = 0.289368 \) is a Hopf bifurcation point for this system. At this point, the linear approximation of (12) possesses a pair of purely imaginary eigenvalues and another pair of stable eigenvalues.

Now, in order to obtain a reduced order model, the centre manifold order is fixed to 3. Equation (6) corresponding to (12) is solved. After substituting stable manifolds for the computed centre manifold, a second order model is obtained for \( \tilde{\mu} = (1 + \varepsilon) \mu_0 \) with \( \varepsilon = 10^{-4} \).

Matlab’s ODE45 solver is used to compute the solutions of the original and reduced models corresponding to \( x(0) = \begin{bmatrix} 10^{-4} & 0 & 0 & 0 \end{bmatrix}^T \). Their dynamic behaviours are plotted in the phase plane and are illustrated in Fig.3. As shown in Fig.3, the reduced model reproduces the same dynamic behaviour as the original model (12). A small error is observed on amplitude of the limit cycle obtained.

### 3.3 Uncertainty propagation

Now, the problem is to analyze short term statistics of the system (12) subject to uncertain initial conditions \( X_1(0) \) assumed to be governed by a uniform distribution law within \( \left[ 10^{-4}, 3.10^{-4} \right] \). The next objective is to estimate the mean values and the standard deviation of the original system responses by using the reduced order model obtained in Subsection 3.2. Monte
Carlo and non intrusive methods are applied to the reduced order model. Before, note the sensitivity of the system responses to dispersions of the initial conditions. This is illustrated in Fig.4 where the phase trajectory \( (X_i(t), \dot{X}_i(t)) \) is plotted for different values of initial condition with the last 200 points.

NISP and regression methods are applied to the reduced model. The Wiener-Legendre expansion of the 5th order is used to estimate the mean value and the standard deviation of the system response in a short term. The results are compared to MC solutions with 1000 uniformly distributed samples within \([10^{-4}, 3.10^{-4}]\).

The mean values of the system responses are plotted in Fig.4 while the standard deviations of \( X_i \) and \( \dot{X}_i \) are plotted respectively in Fig.5 and Fig.6 using the last 200 points. These figures show the precision of the reduced model combined with MC and non-intrusive methods. Indeed, the different combinations well approximate the reference statistics obtained with MC applied to the original model. As we are dealing with short term statistics, accuracy of the reduced model is better than the one observed in the subsection 3.2 where both reduced and original models are simulated for longer time.

The original system responses can be built easily for a given value of the initial conditions using the Wiener-Legendre expansion for which the stochastic modes are computed from the reduced order model. Simulation of the original system is not required. As example, see Fig.7 where the response corresponding to \( x_i(0) = 3.10^{-4} \) is reconstructed.

**4 Conclusion**

This paper has proposed a new methodology to simplify the uncertainty propagation problem in nonlinear dynamic systems. It consists of central manifold theory combined with the polynomial chaos approach. The first method helps to reduce a parameter dependent system in a Hopf bifurcation neighbourhood while the second allows uncertainties in the analysis of nonlinear systems to be taken into account. Moreover, it helps to avoid the prohibitive MC method. The feasibility and efficiency of the proposed methodology has been verified in a simple self-excited mechanism. A third order centre manifold is computed so a second order model is obtained for the considered system. The reduced model reproduces the same dynamic as the original system model which consists of a limit cycle oscillation. In a second step, the reduced model is used to deal with an uncertainty quantification problem. The aim is to estimate the short term statistics of the considered system with respect to uncertain initial conditions. It has been verified that the proposed methodology is an efficient way for the
considered task. The estimation of long term statistics is a more complicated problem due to the polynomial chaos properties which prevent a good estimation of the long term statistics. Other tools can be considered instead of polynomial chaos such as the Haar wavelets expansion. Research work in this context is in progress.

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References:


