Continuum versus Quantum Fields Viewed Through a Scale Invariant Model of Statistical Mechanics

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Abstract: - The implications of a scale invariant model of statistical mechanics to the physical foundations of quantum mechanics and continuum mechanics are described. The nature of the connections between discrete versus continuum fields will be examined. Also, some of the implications of a scale-invariant model of statistical mechanics to the physical foundation of analysis, the continuum hypothesis, and the prime number theory will be addressed.

Key-Words: - Continuum hypothesis, Riemann hypothesis, Goldbach's conjecture, Casimir vacuum, TOE.

1 Introduction

Similarities between stochastic quantum fields [1-17] and classical hydrodynamic fields [18-28] resulted in recent introduction of a scale-invariant model of statistical mechanics [29], and its applications to thermodynamics [30], fluid mechanics [31], and quantum mechanics [32].

In the present study, further implications of the model to the physical foundation of analysis in general and the distribution of spacings between zeros of *Riemann* zeta function and the continuum hypothesis in particular are examined.

2 A Scale Invariant Model of Statistical Mechanics

Following the classical methods [33-37] the invariant definitions of density ρ_β, and velocity of *element* $\mathbf{v}_\beta$, *atom* $\mathbf{u}_\beta$, and *system* $\mathbf{w}_\beta$ at the scale β are given as [30]

$$\rho_\beta = n_\beta m_\beta = m_\beta \int f_\beta du_\beta \quad , \quad \mathbf{u}_\beta = \mathbf{v}_{\beta-1} \qquad (1)$$

$$\mathbf{v}_\beta = \rho_\beta^{-1} m_\beta \int \mathbf{u}_\beta f_\beta d\mathbf{u}_\beta \quad , \quad \mathbf{w}_\beta = \mathbf{v}_{\beta+1} \qquad (2)$$

Similarly, the invariant definition of the peculiar and diffusion velocities are introduced as

$$\mathbf{V}'_\beta = \mathbf{u}_\beta - \mathbf{v}_\beta \quad , \quad \mathbf{V}_\beta = \mathbf{v}_\beta - \mathbf{w}_\beta \qquad (3)$$

such that

$$\mathbf{V}_\beta = \mathbf{V}'_{\beta+1} \qquad (4)$$

For each statistical field, one defines particles that form the background fluid and are viewed as point-mass or "*atom*" of the field. Next, the *elements* of the field are defined as finite-sized composite entities composed of an ensemble of "*atoms*" as shown in Fig.1.

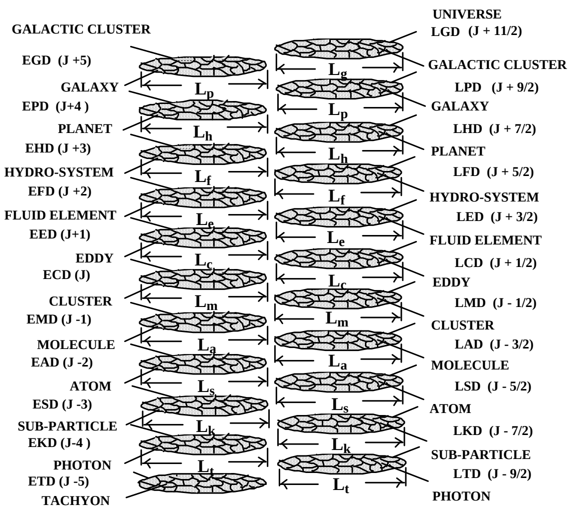

Fig.1 A scale invariant view of statistical mechanics from cosmic to tachyon scales.

Finally, the ensemble of a large number of "*elements*" is defined as the statistical "*system*" at that particular scale.

3 Hierarchies of Statistical Fields from Cosmic to Planck Scales

Clearly, some characteristic scales of space and time will be associated with each member of the hierarchies of embedded statistical fields shown in Fig.1. First, let us start with the field of laminar molecular dynamics LMD when molecules, clusters of molecules (cluster), and clusters of clusters of molecules (eddy) form the "atom", the "element", and the "system" with the velocities $(\mathbf{u}_m, \mathbf{v}_m, \mathbf{w}_m)$. Similarly, the fields of laminar cluster-dynamics LCD and eddy-dynamics LED will have the velocities $(\mathbf{u}_c, \mathbf{v}_c, \mathbf{w}_c)$, and $(\mathbf{u}_e, \mathbf{v}_e, \mathbf{w}_e)$ in accordance with (1)-(2). For the fields of LED, LCD, and LMD, typical characteristic atom, element, and system lengths are [38]

$$\text{EED} \quad (\ell_e, \lambda_e, L_e) = (10^{-5}, 10^{-3}, 10^{-1})\ \text{m} \tag{5a}$$

$$\text{ECD} \quad (\ell_c, \lambda_c, L_c) = (10^{-7}, 10^{-5}, 10^{-3})\ \text{m} \tag{5b}$$

$$\text{EMD} \quad (\ell_m, \lambda_m, L_m) = (10^{-9}, 10^{-7}, 10^{-5})\ \text{m} \tag{5c}$$

If one applies the same (atom, element, system) = $(\ell_\beta, \lambda_\beta, L_\beta)$ relative sizes in (5) to the entire spatial scale of Fig.1, then the resulting cascades or hierarchies of overlapping statistical fields will appear as schematically shown in Fig.2.

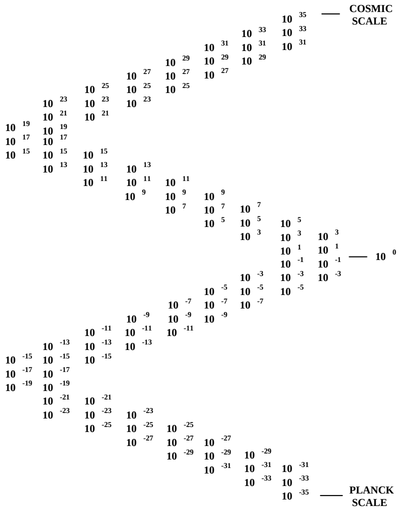

Fig.2 Hierarchy of statistical fields with $(\ell_\beta, \lambda_\beta, L_\beta)$ from cosmic to Planck scales [41].

According to Fig.2, starting from the hydrodynamic scale $(10^3, 10^1, 10^{-1}, 10^{-3})$ after seven generations of statistical fields one reaches the electro-dynamic scale with the element size 10^{-17}, and exactly after seven more generations one reaches the *Planck* length scale $(\hbar G / c^3)^{1/2} \simeq 10^{-35}\,\text{m}$, where G is the gravitational constant. Similarly, exactly seven generations of statistical fields separate the hydrodynamic scale $(10^3, 10^1, 10^{-1}, 10^{-3})$ from the scale of planetary dynamics (astrophysics) 10^{17} and the latter from the scale of cosmology or galactic-dynamics 10^{35} m.

4 Vacuum Fluctuations and Stochastic Natures of Planck and Boltzmann Constants

For all equilibrium statistical fields on the left hand side of Fig.1 since the mean velocity of particle or *Heisenberg-Kramers* virtual oscillator [39] vanishes, the particle energy can be expressed as

$$\varepsilon_\beta = m_\beta \langle u_\beta^2 \rangle = \langle p_\beta \rangle \langle \lambda_\beta^2 \rangle^{1/2} \langle \nu_\beta^2 \rangle^{1/2} \tag{9}$$

where $m_\beta \langle u^2{}_\beta \rangle^{1/2} = \langle p_\beta \rangle$ is the root-mean-square momentum. The result (9) can be expressed in terms of the stochastic *Planck* and *Boltzmann* factors

$$h_\beta = \langle p_\beta \rangle \langle \lambda_\beta^2 \rangle^{1/2} \tag{10}$$

$$k_\beta = \langle p_\beta \rangle \langle \nu_\beta^2 \rangle^{1/2} \tag{11}$$

For the EKD scale one obtains the universal constants of *Planck* [40-41] and *Boltzmann* [28]

$$h = h_k = m_k c \langle \lambda_k^2 \rangle^{1/2} = 6.626 \times 10^{-34}\ \text{J-s} \tag{12}$$

$$k = k_k = m_k c \langle \nu_k^2 \rangle^{1/2} = 1.381 \times 10^{-23}\ \text{J/K} \tag{13}$$

Also, parallel to *de Broglie* hypothesis for the wavelength of matter waves [2]

$$\lambda_\beta = h / p_\beta \tag{14}$$

the frequency of matter waves is given as [28]

$$\nu_\beta = k / p_\beta \tag{15}$$

resulting in the gravitational mass of photon [28]

$$m_k = (hk / c^3)^{1/2} = 1.84278 \times 10^{-41}\ \text{kg} \tag{16}$$

Finite photon mass [42] was anticipated by *Newton* [43] and is in accordance with *Einstein-de Broglie* theory of light [44-46]. *Avogardo-Loschmidt* number is identified as [28]

$$N^o = 1/(m_k c^2) = 6.0376 \times 10^{23} \tag{17}$$

that leads to the modified universal gas constant

$$R^o = N^o k = 8338 \quad \text{J/(kmol.K) = J/kcal} \tag{18}$$

The modified universal gas constant R^o was recently identified [47] as *De Pretto* number 8338 J/kcal that appeared in the mass–energy equivalence equation of *De Pretto* [48]

$$E = mc^2 = mc^2 / 8338 \tag{19}$$

A most significant implication of the model in Figs.1 and 2 concerns the nature of physical space, vacuum, that is identified as a *tachyonic fluid* that is the stochastic ether of *Dirac* [49]

"We can now see that we may very well have an aether, subject to quantum mechanics and conforming to relativity, provided we are willing to consider the perfect vacuum as an idealized state, not attainable in practice. From experimental point of view, there does not seem to be any objection to this. We must make some profound alterations in our theoretical ideas of the vacuum. It is no longer a trivial state, but needs elaborate mathematics for its description."

It is emphasized that *space* is the *tachyonic fluid itself* and not *merely a container that is occupied by* this fluid, as in the classical theories of ether [50]. Using a glass of water as an example, the physical space is analogous to the water itself, and not to the glass.

In the ontology of universe according to *Newton* [43], *absolute* space was considered to be a *non-participating container* filled with medium called *ether* in order to describe the phenomena of light as well as gravitation [43]

"Qu.19. Doth not the refraction of light proceeds from different density of this aethereal medium in different places, the light receding always from the denser parts of the medium? And is not the density thereof greater in free space and open spaces void of air and other grosser bodies, than within the pores of water, glass, crystal, gems, and other compact bodies? For when light passes through glass or crystal, or falling very obliquely upon the farther surfaces thereof is totally reflected, the total reflection ought to proceed rather from the density and vigor of the medium without and beyond the glass, than from the rarity and weakness thereof.

Qu.22. May not planets and comets, and all gross bodies, perform their motions more freely, and with less resistance in this aethereal medium than in any fluid, which fills all space adequately without leaving any pores, and by consequence is much denser than quick-silver or gold ? And may not its resistance be so small, as to be inconsiderable? For instance: if this aether (for so I will call it) should be supposed 700,000 times more elastic than our air, and above 700,000 times more rare, its resistance would be above 600,000,000 times less than of water. And so small a resistance would scarce make any sensible alteration in the motions of the planets in ten thousand years. If any one would ask how a medium can be so rare, let him tell me how the air, in the upper parts of the atmosphere, can be above hundred thousand thousand times rarer than gold."

The atmosphere of opposition by scientists against the hypothesis of ether was most eloquently described by *Maxwell* in his treatise [51]:

"There appears to be, in the minds of these eminent men, some prejudice, or a priori objection, against the hypothesis of a medium in which the phenomena of radiation of light and heat and the electric action at a distance take place. It is true that at one time those who speculated as to the cause of the physical phenomena were in the habit of accounting for each kind of action at a distance by means of a special aethereal fluid, whose function and property it was to produce these actions. They filled all space three and four times over with aethers of different kinds, so that more rational inquirers were willing rather to accept not only Newton's definite law of attraction at a distance, but even the dogma of Cotes, that action at a distance is one of the primary properties of matter, and that no explanation can be more intelligible that this fact. Hence undulatory theory of light has met with much opposition, directed not against its failure to explain the phenomena, but against its assumption of the existence of a medium in which light is propagated."

The existence of the medium called *ether* was found to be indispensable for the proper description of electrodynamics according to *Lorentz* [52]

" I cannot but regard the ether, which can be the seat of an electromagnetic field with its energy and its vibrations, as endowed with certain degree of substantiality, however different it may be from all ordinary matter,"

The participation of ether in the transmission of perturbations as well as the possible granular structure of space were anticipated by *Poincaré* [53]

"We might imagine for example, that it is the ether which is modified when it is in relative motion in reference to the material medium which it penetrates, that when it is thus modified, it no longer transmits perturbations with the same velocity in every direction."

Also, the notion of ether was considered by *Einstein* as not only consistent with the General Theory of Relativity, but in his opinion according to GTR space without ether is *unthinkable* [54]

"Recapitulating, we may say that according to the general theory of relativity space is endowed with physical qualities; in this sense, therefore, there exists an ether. According to the general theory of relativity space without ether is unthinkable; for in such space there not only would be no propagation of light, but also no possibility of existence for standards of space and time (measuring-rods and clocks), nor therefore any space-time interval in the physical sense. But this

ether may not be thought of as endowed with the quality characteristic of ponderable media, as consisting of parts which may be tracked through time. The idea of motion may not be applied to it."

The statement "space without ether" shows that ether was considered as a medium that *filled the space* rather than *being the space* itself. Also, because stochastic *Planck* and *Boltzmann* constants (14)-(15) relate to *vacuum fluctuations*, contrary to the above statement by *Einstein*, the idea of *rest* rather than *motion* may not be applied to the ether. That is, *Dirac's* stochastic ether cannot satisfy both the principles of relativity and quantum mechanics if it is at rest. Ironically, parallel to static rather than dynamic vacuum at *Planck* scale, *Einstein* also chose a static rather than dynamic universe at cosmic scale (see Fig.2) that resulted in his introduction of the cosmological constant.

Recently, a modified *van der Waals* equation of state was introduced [55] leading to a finite pressure of vacuum in harmony with the finite zero-point energy of *Casimir* vacuum [56]. Hence, the pressure of matter p_m and anti-matter p_{am} fields will be respectively larger and smaller than the vacuum pressure p_v [55]

$$0 = p_{WH} < p_{am} < p_v < p_m < p_{BH} = \infty \qquad (20)$$

and limited by the pressures of *white hole* $p_{WH} = 0$ and *black hole* $p_{BH} = \infty$ as two singularities of the field. Hence, the concept of "empty" space in the following statement by *Hawking* [57]

"Maxwell's theory predicted that radio or light waves should travel at a certain speed. But Newton's theory had got rid of the idea of absolute rest, so if light was supposed to travel at a fixed speed, one would have to say what that fixed speed was to be measured relative to. It was therefore suggested that there was a substance called "ether" that was present everywhere, even in "empty" space. Light waves should travel through the ether as sound waves travel through air, and their speed should therefore be relative to the ether."

, corresponds to *Casimir* vacuum [56] and not to the white hole that is a singularity of the field where the fabric of space has been ruptured.

Physical space is considered to be a *compressible fluid* [58] in harmony with compressible ether model of *Planck* [59]. Therefore, parallel to atmospheric air that becomes compressible when the *Mach* number Ma = v/a approaches unity, the tachyonic fluid that constitutes the physical space becomes compressible when the *Michelson* number defined as Mi = v/c approaches unity, with a and c denoting the velocity of sound and light, respectively. Thus, Ma >1 (Mi >1) corresponds to *supersonic* (*superchromatic*) flow, leading to *Mach* (*Poincaré-Minkowski*) cone that separates the zone of *sound* (*light*) from the zone of *silence* (*darkness*) [58].

The compressibility of space was recently shown to result in *Lorentz-FitzGerald* contraction [58] thereby providing a *causal* explanation [60] of relativistic effects [61] in harmony with the perceptions of *Poincaré* and *Lorentz* [50]. Within the compressible physical space, dark energy, dark matter, and baryonic matter are all considered to be composed of photons [58] and hence could interchange in harmony with the *metamorphosis* of matter into radiation and vice-versa first recognized by *Newton* [43]

"Qu.30. Are not gross bodies and light convertible into one another, and may not bodies receive much of their activity from the particles of light which enter their composition? The changing of bodies into light, and light into bodies, is very comfortable to the course of Nature, which seems delighted with transmutations. And among such various and strange transmutations, why may not Nature change bodies into light and light into bodies?"

5 Universal and Scale Invariant Foundation of Quantum Mechanics

According to a recent study [32], the energy spectrum of all isotropic equilibrium statistical fields shown in Fig.1 are governed by the invariant *Planck* energy distribution law [40]

$$\frac{\varepsilon_\beta dN_\beta}{V} = \frac{8\pi h}{u_\beta^3} \frac{\nu_\beta^3}{e^{h\nu_\beta/kT} - 1} d\nu_\beta \qquad (21)$$

At large wave numbers, the universal *Planck* law (21) gives the *Kolmogorov* $\kappa^{-5/3}$ law [32]. The energy of each oscillator is $\varepsilon_\beta = h\nu_\beta$. Preliminary examinations of the three-dimensional energy spectrum E(k, t) for isotropic turbulence appear to support such a correspondence [62, 63].

Under thermodynamic equilibrium, the speed of particles will be governed by the invariant *Maxwell-Boltzmann* distribution function

$$\frac{dN_{u\beta}}{N} = 4\pi \left(\frac{m_\beta}{2\pi k T_\beta}\right)^{3/2} u_\beta^2 \, e^{-m_\beta u_\beta^2/2kT_\beta} du_\beta \qquad (22)$$

that was derived directly from the invariant *Planck* distribution function [32]. By (22), one arrives at a hierarchy of embedded *Maxwell-Boltzmann* distribution functions for …EED, ECD, and EMD scales as shown in Fig.3.

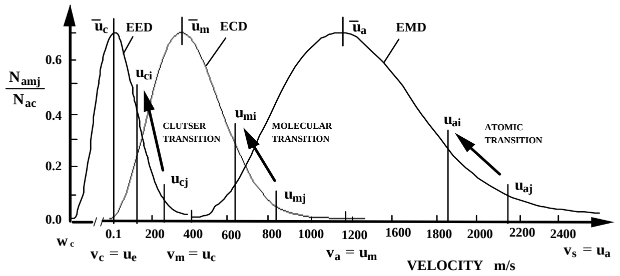

Fig.3 Maxwell-Boltzmann speed distribution viewed as stationary spectra of cluster sizes for EED, ECD, and EMD scales at 300 K [71].

The invariant conservation equations for an incompressible and irrotational $\mathbf{v}_\beta = \nabla \Phi_\beta$ flow with the velocity potential Φ_β lead to the invariant *Bernoulli* equation [32]

$$\frac{\partial(\rho_\beta \Phi_\beta)}{\partial t} + \frac{(\nabla \rho_\beta \Phi_\beta)^2}{2\rho_\beta} + p_\beta = \text{cons}\tan\text{t} = 0 \qquad (23)$$

Comparison of (23) with the *Hamilton-Jacobi* equation of classical mechanics [2] resulted in the introduction of the invariant action and quantum mechanic wave function as [32, 64]

$$S_\beta(\mathbf{x}, t) = \rho_\beta \Phi_\beta \quad , \quad \Psi_\beta(\mathbf{x}, t) = S'_\beta(\mathbf{x}, t) \qquad (24)$$

The wave function (24) was recently applied [32] to derive from (23) the invariant time-dependent *Schrödinger* equation [65]

$$i\hbar_{o\beta} \frac{\partial \Psi_\beta}{\partial t} + \frac{\hbar_\beta^2}{2m_\beta} \nabla^2 \Psi_\beta - \bar{U}_\beta \Psi_\beta = 0 \qquad (25)$$

It is now clear that the potential energy $\bar{U}_\beta$ [32] in (25) acts as *Poincaré stress* [66-69] and is responsible for the stability of "particles" or *de Broglie* wave packets.

Soon after the introduction of his equation, *Schrödinger* himself [70] tried to identify some type of ensemble average interpretation of the wave function. Clearly, according to (24) the statistical ensemble nature of Ψ_β naturally arises from the velocity potential Φ_β. To reveal the universal (Fig.1) physical foundation of quantum mechanics one starts by noting that the statistically stationary size of "clusters" is governed by the *Maxwell-Boltzmann* distribution function (22) [32]. Next, one views the transfer of a cluster from a small rapidly oscillating eddy (j) to a large slowly oscillating eddy (i) as transition from the high energy level (j) to the low energy level (i), see Fig.3, as schematically shown in Fig.4,

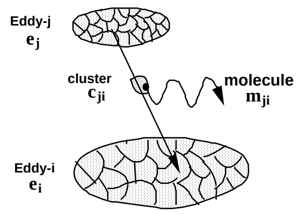

Fig.4 Transition of cluster c_{ij} from eddy-j to eddy-i leading to emission of molecule m_{ij}.

Such transitions (Fig.4) will be accompanied with emission of a "sub-particle" or "molecule" to carry away the excess energy [71]

$$\Delta\varepsilon_{ji\beta} = \varepsilon_{j\beta} - \varepsilon_{i\beta} = h(\nu_{j\beta} - \nu_{i\beta}) \qquad (26)$$

in harmony with *Bohr's* theory of *atomic* spectra [39]. Therefore, the reason for the *quantum nature* of "molecular" energy spectra in equilibrium isotropic turbulent fields is that transitions must occur between eddies whose energy levels must satisfy the criterion of *stationarity* imposed by the *Maxwell-Boltzmann* distribution function [71].

The hydrodynamic origin of *Schrödinger* equation (23)-(25) and the stochastic models of electrodynamics [1-17] are in harmony with the quantum mechanical foundation of turbulence (21) and (25). At the *Planck* scale $(\hbar G / c^3)^{1/2} \simeq 10^{-35}$ m (see Fig.2) the hydrodynamic field represents the physical space [71] or the *Casimir* vacuum [56] with its fluctuations. Also, at cosmic scales 10^{35} with β = g (Figs.1, 2), the wave function Ψ_g will correspond to the wave function of the universe [72, 73]. The wave-particle duality of galaxies has been established by their observed quantized red shifts [74]. Also, the dual and seemingly incompatible objective versus subjective natures of Ψ_β, emphasized by *de Broglie* [1-3], can now be resolved. This is because the objective part of Ψ_β (24) is associated with the density ρ_β and accounts for the particle localization while the subjective part of Ψ_β is associated with the complex velocity potential Φ_β that accounts for the observed action-at-a-distance as well as renormalization and thereby the success of *Born's* [75] probabilistic interpretation of Ψ_β.

6 Invariant Definitions of System, Element, and Atomic Lengths and Times

The spatial distance of each statistical field in Fig.1 is measured on the basis of the number of "atoms" of that particular statistical field $N_{A\beta}$ [29, 76]

$$x'_\beta = \ln N_{A\beta} \qquad (27)$$

With the definition (27) counting of numbers must begin with the number zero naturally since it corresponds to one atom. The characteristic lengths of (system, element, atom) = $(L'_\beta, \lambda_\beta, \ell_\beta)$ at scale β are defined as

$$L'_{\infty\beta} = \ln N_{AS\beta}$$
$$\lambda_\beta = \ln N_{AE\beta} \tag{28}$$
$$\ell_\beta = 0_\beta \quad , \quad N_{A\beta} = 1$$

where $(N_{AS\beta}, N_{AE\beta})$ respectively refer to the number of atoms in the system and the element. In view of the definitions in (27)-(28), the following schematic representation of the hierarchy of length scales may be obtained

. . .

$$L'_{\infty\beta+1} \,\rule{3cm}{0.4pt}\, \lambda_{\beta+1} \,\rule{1.5cm}{0.4pt}\, 0_{\beta+1}$$
$$L'_{\infty\beta} \,\rule{1.5cm}{0.4pt}\, \lambda_\beta \,\rule{0.7cm}{0.4pt}\, 0_\beta$$

. . .

(29)

The size of the element and atom of adjacent statistical fields within the cascade (Fig.1) are related as

$$N_{AE\beta} = N_{AS\beta-1} \tag{30}$$
$$N_{A\beta} = N_{AE\beta-1} \tag{31}$$

hence

$$\lambda_\beta = L'_{\infty\beta-1}$$
$$0_\beta = \lambda_{\beta-1} \tag{32}$$

The total number of atoms of the system will be given by

$$N_{AS\beta} = \left(N_{AE\beta}\right)^{N_{ES\beta}} \tag{33}$$

By (27) and (33), the system length at scale β becomes

$$L'_{\infty\beta} = \ln N_{AS\beta} = N_{ES\beta} \ln(N_{AE\beta}) \tag{34}$$

The element length λ_β will be defined as the unit or "measure" such that by (27) and (28) one can introduce the "dimensionless" or "measureless" coordinate [76]

$$x_\beta = \frac{x'_\beta}{\lambda_\beta} = \frac{\ln N_{A\beta}}{\ln N_{AE\beta}} = N_{ES\beta} \tag{35}$$

Therefore, the natural numbers 1, 2, 3, … of a *Euclidean* space at scale β refer to integral numbers of this *constant* "measure" λ_β defined in (28). With the "measureless" coordinate (35) the cascade (29) will become *normalized* [29]

. . .

$$L_{\infty\beta} \,\rule{3cm}{0.4pt}\, 1_\beta \,\rule{1.5cm}{0.4pt}\, 0_\beta$$
$$L_{\infty\beta-1} \,\rule{0.8cm}{0.4pt}\, 1_{\beta-1} \,\rule{0.5cm}{0.4pt}\, 0_{\beta-1}$$

. . .

(36)

It is interesting to examine the connection of the result (35) with the classical definition of fractal dimension [77, 78]

$$D = -\frac{\ln N(r)}{\ln(r)} \tag{37}$$

where N(r) is the total number of unit shapes and r is the size of the coarse–graining. Hence, one can introduce a scale invariant definition of fractal dimension as [76]

$$D_\beta = N_{AE\beta} = N_{ES\beta-1} = \frac{\ln N_{AS\beta-1}}{\ln(N_{AE\beta-1})} \tag{38}$$

that by (37) leads to the invariant definition of coarse-graining

$$r_\beta = \frac{1}{N_{AE\beta-1}} \tag{39}$$

Since each element could contain very large number of "atoms", the fractal dimensions of typical statistical field shown in Fig.1 could be exceedingly large 10^7.

The above definitions for the length $(L_\beta, \lambda_\beta, \ell_\beta)$ and velocity $(\mathbf{w}_\beta, \mathbf{v}_\beta, \mathbf{u}_\beta)$ result in the following definitions of the system, element, and atomic "time" $(\Theta_\beta, \tau_\beta, t_\beta)$ for the statistical field at scale β [29]

$$\Theta_\beta = L_\beta / w_\beta = \tau_{\beta+1} \tag{40a}$$
$$\tau_\beta = \lambda_\beta / v_\beta = t_{\beta+1} \tag{40b}$$
$$t_\beta = l_\beta / u_\beta = \tau_{\beta-1} \tag{40c}$$

Thus, one arrives at a hierarchy of *time elements* for the statistical fields shown in Fig.1 as

$$\ldots\ \tau_e > \tau_c > \tau_m > \tau_p > \tau_s > \ldots \tag{41}$$

Clearly, the most fundamental and universal physical time is the time associated with the tachyon fluctuations $\tau_t = t_k$ [71] of *Casimir* vacuum [56] at the *Planck* scale in ETD field shown in Fig.1.

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7 Implications to the Distribution of Spacings between Zeros of Riemann Zeta Function

The objectives of the previous sections 2-7 were in part to prepare the grounds to address the implications of the scale invariant model of statistical mechanics (Fig.1) to the mathematical problem of the existence of infinitesimals and *Hilbert's* number eight problem namely the *Riemann* hypothesis.

The classical problem of the existence of infinitesimals within the framework of nonstandard analysis of *Robinson* [79] and *Internal Set Theory* IST of *Nelson* [80] was recently discussed through the introduction of "dimensionless" or "measureless" numbers [76]

$$x_\beta = \frac{x'_\beta}{\lambda_\beta} = N_{ES\beta} \tag{42}$$

with the "measure" λ_β defined as

$$\lambda_\beta = L_{+\infty\beta-1} = \int_{0_{\beta-1}}^{\infty_{\beta-1}} e^{-x_{\beta-1}^2} dx_{\beta-1} = \frac{\sqrt{\pi_{\beta-1}}}{2} \tag{43}$$

The measure (43) has been chosen on the basis of *Gauss's* error function on account of the equilibrium, i.e. random, distribution of the particles (Fig.1). By (42)-(43), the range $(-1_\beta, 1_\beta)$ of the outer coordinate x_β will correspond to the range $(-\infty_{\beta-1}, \infty_{\beta-1})$ of the inner coordinate $x'_{\beta-1}$ leading to the coordinate hierarchy schematically shown in Fig.5.

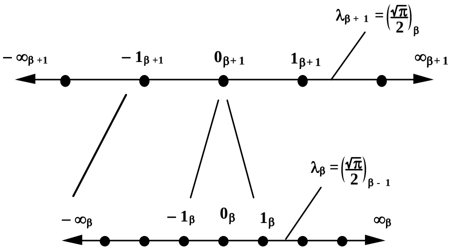

Fig.5 Hierarchy of normalized coordinates for cascades of embedded statistical fields [76].

The model shown in Fig.5 involves the concepts of "point", "aggregate", and "infinity" that are all central to the problem of continuum and require exploration. First, it is interesting to explore possible connections between the random distributions of numbers on the line (Fig.5) in mathematics versus the distribution of particles amongst various energy levels in physics (Fig.3). The hydrodynamic model of *Schrödinger* equation (25) and the stochastic models of electrodynamics [1-17] provide a new paradigm for the physical foundation of quantum mechanics as discussed in section 6. According to the new paradigm, the energy spectra given by *de Broglie-Schrödinger* wave mechanics or *Heisenberg* [81] matrix mechanics will correspond to different size clusters, *de Broglie* wave packets [71] as shown in Fig.3 with the associated transitions shown in Fig.4.

Some of the implications of the new paradigm mentioned above to the important problem of *Riemann Hypothesis* will now be examined. In their recent works *Montgomery* [82] and *Odlyzko* [83] connected the normalized spacings between the zeros of *Riemann* zeta function to the normalized spacings between the eigenvalues of Gaussian Unitary Ensemble GUE resulting in what is known as *Montgomery-Odlyzko* law [84, 85]. The pair correlation of *Montgomery* [82] was subsequently recognized by *Dyson* to correspond to that between the energy levels of heavy elements [84, 85] and thus to the pair correlations between eigenvalues of *Hermitian* matrices [86]. Hence, a connection was established between quantum mechanics on the one hand and quantum chaos [87] on the other hand.

According to Figs.1 and 3, each atom of the EED field will be composed of an ensemble of molecular clusters and as such also represents an equilibrium molecular dynamic EMD system. Therefore, at thermodynamic equilibrium each point of the external field EED must be in thermal equilibrium with the internal field EMD. Thus, one arrives at a hierarchy of embedded equilibrium statistical fields whose particles must satisfy the invariant *Maxwell-Boltzmann* speed distribution (22) as shown in Fig.3. The energy spectrum of all such equilibrium statistical fields will be governed by the invariant *Planck* energy distribution law (21) [32].

The comparison between normalized *Maxwell-Boltzmann* distribution function

$$f(u_\beta) = 4\pi \left(\frac{B^2 m_\beta}{2\pi k T_\beta}\right)^{3/2} u_\beta^2 \; e^{-m_\beta (B u_\beta)^2 / 2kT_\beta} \tag{44}$$

and the distribution of normalized spacings between zeros of *Riemann* zeta function γ_n starting from n = 10^{12} from calculations of *Odlyzko* [83] are shown in Fig.6. The distribution (44) shown in Fig.6 corresponds to nitrogen gas $m = 28 \times 1.66 \times 10^{-27}$ kg at the temperature of T = 380 K with the factor B = $v_\beta = (2kT/m)^{1/2} = 534$ for normalization when speed is made dimensionless through division with the most probable speed $v_\beta = u_{\beta+1}$.

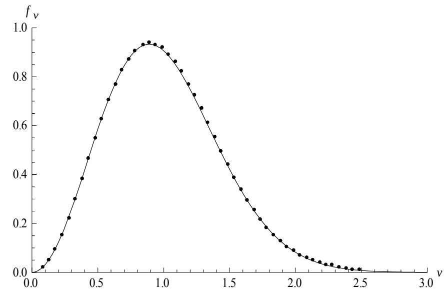

Fig.6 Comparison of distributions of the normalized spacings between zeros of zeta function $\gamma_n, 10^{12} \le n \le 10^{12} + 10^5$ [83] and the normalized Maxwell-Boltzmann distribution (44).

Since the exact connections between the mathematical and the physical theories could be addressed *a posteriori*, at this point the physical parameters associated with mass m, *Boltzmann* constant k, and temperature T in (44) are removed. Furthermore, instead of the cluster speed that is related to the cluster size of particles one considers the cluster size of numbers, prime number aggregates or condensations, and hence introduces the coordinate x'_β (35)

$$x'_{\beta i} = \ln N_{AS\beta i} = N_{ES\beta i} \ln(p_{\beta i}) \qquad (45)$$

Since particle speed is related to the square root of particle energy $v_\beta \propto (\overline{u_\beta^2})^{1/2}$ it is expected that the requirement of having ½ as the real part of the zeros of *Riemann* zeta function is somehow related to this power of ½ between speed and energy.

Thus, one introduces a Normalized Maxwell-Boltzmann NMB distribution in the form

$$\rho_i = (8/\pi_\beta)\, [(2/\sqrt{\pi_\beta})x_{\beta i}]^2 \; e^{-[(2/\sqrt{\pi_\beta})x_{\beta i}]^2} \qquad (46)$$

where the coordinate (45) is made "dimensionless" through division by the mean cluster size

$$x_{p\beta i} = x'_{p\beta i} / \overline{N}_{\beta p i} = (N_{\beta p i} / \overline{N}_{\beta p i}) \ln p_{\beta i} \qquad (47)$$

and the additional division by the "measure" $\sqrt{\pi_\beta}/2$ in (46) is for normalization [76] in accordance with Fig.5. Direct comparisons between NMB in (46) and the normalized spacings between the zeros of the *Riemann* zeta function and the eigenvalues of GUE from [83] are shown in Fig.7. Although the coordinate (47) is discrete (quantized $N_{\beta i}$) because the number of "atoms" in the mean-sized cluster $\overline{N}_{\beta i}$ is considered to be very large, the distribution function will look smooth as shown in Fig.7.

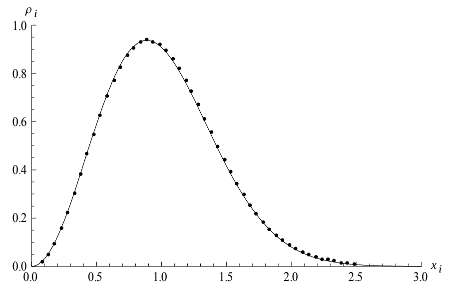

Fig.7 Probability density of normalized spacings between zeros of Riemann zeta function $\gamma_n, 10^{12} \le n \le 10^{12} + 10^5$ [83], normalized spacings between eigenvalues of GUE [83], and the NMB distribution (46).

Because each partial density in (46) corresponds to single specie (prime), one next constructs a "mixture" density by adding all the NMB distributions for partial densities of all species

$$\rho = \sum_i \rho_i \qquad (48)$$

$$x_\beta = \sum_i x_{\beta i} = (N_\beta / \overline{N}_{\beta t}) \ln \prod_i p_{\beta i} \qquad (49)$$

with $N_\beta / \overline{N}_{\beta t} = N_{\beta i} / \overline{N}_{\beta t i}$. The mixture probability density $\rho(x_\beta)$ will have the same form as (46) with NMB distribution as shown in Fig.7. Such a grand ensemble of NMB $p_{\beta i}$-adic statistical fields will lead to a corresponding GUE that could be identified as *Connes'* Adele space [84, 85] $\mathfrak{A}_\beta$ for a particular scale β. Because such a GUE is based on p-adic type numbers (47), the normalized spacings between its eigenvalues should be related to the normalized spacings between the zeros of *Riemann* zeta function in accordance with the predictions of noncommutative geometry of *Connes* [88].

At any scale π_β defining unity (Fig.5) and with $p_{\beta i}$ as "atomic" species of hierarchies of statistical fields (Fig.1), there are two ways of constructing an even number with relevance to *Goldbach's* conjecture [89]. First, are the homogeneous even numbers N_{EH} made of pairs of identical atoms

$$N_{EH} = p_{\beta i} + p_{\beta i} = 2p_{\beta i} \qquad (50)$$

such as 4 = 2+2, 6 = 3+3, 10 = 5+5, … . Next, are the non-homogeneous even numbers N_{EN} made of pairs of two different types of atomic species

$$N_{EN} = (p_{\beta i} + p_{\beta j}) + (p_{\beta i} + p_{\beta j}) = 2(p_{\beta i} + p_{\beta j}) \qquad (51)$$

But since $(p_{\beta I} + p_{\beta j})$ is even $N_{EN} = 2N_{EN1/2}$ (51) gives

$$N_{EN1/2} = p_{\beta i} + p_{\beta j} \qquad (52)$$

such as 8 = 3 +5, 12 = 5+7, …, that unlike (50) are no longer separable into atomic identical twins.

If one considers in the spirit of *Pythagoras* and *Plato* that pure "numbers" are the basis of all that is physically "real", then these "atomic" prime numbers applied to construct each p-adic statistical field and their associated p-adic matrices may lie at the foundation of *Riemann Hypothesis* in harmony with noncommutative geometry [88]. That is, when the *physical space* itself, the *Casimir* vacuum [56], is identified as a *fluid* governed by a statistical field [71, 90], it will have a spectrum of energy levels given by the *Schrödinger* equation (25) that in view of *Heisenberg* [81] matrix mechanics will be described by noncommutative geometry [88]. Although the exact connection between noncommutative geometry and the *Riemann* hypothesis is yet to be understood according to *Connes* [85]

"The process of verification can be very painful: one's terribly afraid of being wrong…it involves the most anxiety, for one never knows if one's intuition is right- a bit as in dreams, where intuition very often proves mistaken"

the model suggested above may help in the construction of the physical foundation of such a mathematical theory.

8 Implications to the Continuum Hypothesis

In this section, some of the implications of the model, Fig.1, to *Hilbert's* first problem namely the important problem of continuum hypothesis [89] will be presented. Historical development of the concept of continuum has been reviewed in an excellent book by *Bell* [91]. In the following, it will be shown that the results of the present field theory discussed in sections 2-7 will facilitate the understanding of the various concepts of the continuum hypothesis such as "points", "aggregates", and "infinity" discussed by *Bell* [91].

The most central concept of continuum hypothesis is that of the "atom" or mathematical "point". As seen in Figs. 1 and 2, the definition of point most naturally depends on the scale being considered such that

$$(\text{Atom})_\beta = (\text{Element})_{\beta-1} = (\text{System})_{\beta-2} \qquad (53)$$

Empirical observations across the entire spatial scales shown in Fig.1 suggest that there are no limits of either infinitely large $\beta > g$ or infinitely small $\beta < t$. This is in harmony with the philosophy of *Anaxagoras* who said [91]

"Neither is there a smallest part of what is small, but there is always a smaller, for it is impossible that what is should ever cease to be"

The above statement could also be made about the impossibility of existence of the largest part of what is large if with *Newton* one conceives that the universe is infinite.

While atoms are assumed as points, elements are considered to be extended in accordance with the perceptions of *Poincaré* [91]

"we cannot say that our element is without extension, since we cannot distinguish it from neighboring elements and it is thus surrounded by a sort of haze. If the astronomical comparison may be allowed, our 'elements' would be like nebulae, whereas the mathematical points would be like stars"

The above statement exactly corresponds to the scale of equilibrium planetary dynamics EPD $\beta = p$ in Fig.1 where atoms correspond to planets.

The objection emphasized by *Bois-Reymond* [91] as to how to construct a finite extension from collection of many points that are devoid of size is now resolved by the definition of point in (53), besides the fact that points, "atoms", are considered to be under constant motion except at the absolute zero temperature T = 0. Also, the invariant fractal definition of point resolves the objection of *Democritus* [92]

"It is contended that division is possible; very well, let it be performed. What remains? No bodies; for these could be divided still further, and the division would not have progressed to the ultimate stage. There could only be points, and the body would have to be composed of points, which is evidently absurd"

Since as stated by *Anaxagoras* it is impossible that what is should ever cease to be, the possibility of *Aristotle's* potentially infinite division cannot be ruled out just because of limitation of available energy or human imagination.

Similar considerations also apply to the mathematical theory of time in describing the temporal continuum discussed by *Weyl* [93]

"Exact time- or space-points are not the ultimate, underlying, atomic elements of the duration or extension given to us in experience. On the contrary, only reason, which thoroughly penetrates what is experientially given, is able to grasp exact ideas"

From the invariant definitions of atomic, element, and system times in (40) [29]

$$t_\beta = \tau_{\beta-1} = \Theta_{\beta-2} \quad , \qquad (54)$$

one can form a finite time duration by addition of many instants of time and write

$$\tau_\beta = \Sigma t_\beta = \Sigma \tau_{\beta-1} \qquad (55)$$

Parallel to *Heisenberg* [81] spatial uncertainty principle

$$\Delta\lambda_\beta \ \Delta p_\beta \geq h \qquad (56)$$

that limits the resolution of spatial measurements, there is the *temporal uncertainty principle* [71]

$$\Delta\nu_\beta \ \Delta p_\beta \geq k \qquad (57)$$

that limits the resolution of time measurements.

Another significant feature of continuum is that of the element, aggregate or condensation of points [91]. According to the physical model, the formation and stability of atomic aggregates, clusters, is due to *Poincaré* stress in the invariant *Schrödinger* equation (25). The formation of numerical point aggregates in mathematics is related to the formation of atomic clusters, *de Broglie* wave packets, in physics through the connection between distribution of normalized spacing between zeros of *Riemann* zeta function and NMB distribution shown in Fig.6-7 as discussed above.

While *Weyl* suggests that [93]

"A 'hierarchical' version of analysis is artificial and useless. It loses sight of its proper object, i.e. number",

the results presented in sections 2-7 prove otherwise. Indeed, the hierarchical model shown in Fig.1 is both natural and harmonious with empirical observations and provides a useful visual and geometrical tool for the description of the concepts of continuum, point, limit, and infinite. For example, the problem of *Anaxagoras* that a continuum cannot be composed of discrete elements [93]

"chopped off from one another, as it were, with a hatchet",

could be resolved by the hierarchical model of coordinates shown in Fig.5 while addressing *Aristotle's* concern [92]

"if the continuous line is divided into two halves, the one dividing point is taken for two; it is both beginning and end . But as one divides in this manner, neither the line nor the motion are any longer continuous …In the continuous there is indeed an unlimited number of halves, but only in possibility, not in reality"

According to Fig.5, specification of the exact location of the cut to an infinite degree of accuracy requires one to decompactify the finite intervals $(0\ ,\ 1\ )_\beta$ to infinitely smaller scales $(0\ ,\ 1\ )_{\beta-1}$, $(0\ ,\ 1\ )_{\beta-2}$, … within the hierarchy.

It appears that in Fig.5 finite completed intervals have been constructed by adding an infinite number of elements thus contradicting *Gauss* [94].

"I object to the use of an infinite magnitude as something completed; this is never admissible in mathematics. One must not interpret infinity literally when, strictly speaking, one has in mind a limit approached with arbitrary closeness by ratios as other things increase without bounds."

However, because of the transcendental nature of the number π_β occurring in (43) the intervals are not strictly speaking ever completed.

The transcendental numbers constituting an uncountable subset of any interval play a central role in *Cantor's* [95] concept of completed infinite discussed by *Casti* [89]

"Put another way, Cantor knew there were far more real numbers than could be accounted for by the relatively small set of algebraic numbers. Where did all these real numbers come from? The only thing they could be is the mysterious transcendental numbers. So, like the postulated "dark matter" of the universe that physicists believe constitutes the mass needed to keep the universe from flying off to infinity, the transcendental numbers keep real numbers "together," so to speak- even though Cantor could not find a single, concrete example of one of them."

In view of Fig.5, one requires $\aleph_{o\beta}$ to count numbers on the interval $(0\text{-}1)_\beta$ at a given number of digits of the transcendental measure λ_β (43), and $\aleph_{1\beta}$ to count the interval with the addition of the next digit to the transcendental measure λ_β, and so on at infinitum. Hence, to count the numbers of all the intervals of all scales one requires

$$\aleph_{o\beta},\ \aleph_{1\beta},\ \aleph_{2\beta}\ \ldots\ ,\ \aleph_{o\beta+1}, \aleph_{1\beta+1},\ \aleph_{2\beta+1}\ \ldots\ , \qquad (58)$$

thus suggesting that *Cantor's* [95] *Continuum Hypothesis* is not true and there exist continua of first, second, … orders $C_1,\ C_2, \ldots C_\infty$ in harmony with the perceptions of *Poincaré* and *Gödel* [89, 91]. Also, the hierarchies of turbulent fields in Fig.1 suggest that *Gödel's* incompleteness theorem be viewed as the *closure problem* of turbulence [76]. That is, specification of the exact state of the system at any scale β is impossible because of the external effects of the adjacent scales β+1 and β-1 from both boundaries without and within the system. Although the physical model presented herein may lead to modification of the ZF axiomatic system, the result (55) is harmonious with the undecidability of the *Continuum Hypothesis* [89, 96].

9 Concluding Remarks

Some implications of a scale invariant statistical theory of fields to *Riemann* and continuum

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hypotheses were described. In particular, it was found that the distribution of the normalized spacings between the zeros of *Riemann* zeta function follow the normalized *Maxwell-Boltzmann* distribution function. Also, the results suggest that *Cantor's* continuum hypothesis is not true. The universal nature of hierarchies of statistical fields across broad range of spatio-temporal scales is in harmony with the observed universal occurrence of fractals in physical science emphasized by *Takayasu* [58].

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