Spectral Method for Solving Integral-Differential Equations with an Integral Condition

NIKOS MASTORAKIS
Technical University of Sofia,
BULGARIA
http://www.wseas.org/mastorakis

OLGA MARTIN
University “Politehnica” of Bucharest
ROMANIA
omartin_ro@yahoo.com

Abstract: In this paper a method has been proposed to replace an integral-differential equation with a partial differential equation of second order. This is accompanied by a boundary problem and a nonlocal condition. Its solution, which is a trigonometric series with variable coefficients, will be obtained using superposition principle and a spectral method. Comparative study of the solution with the exact solution for two numerical examples will lead to the conclusions regarding the values of errors.

Key words: Stationary transport equation, integral-differential equation, spectral method, finite difference method.

1 Introduction

The spectral method that we use in our paper is closely related with the finite element method (FEM). The main difference between them is that FEM approximates the solution as a linear combination of piecewise functions, which are nonzero on small subdomains, while a spectral method approximates the solutions as linear combination of continuous functions that are generally nonzero over the domain of solution. Therefore, the finite element method is a local approach, while the spectral method is a global approach of the phenomenon. The basic idea of the algorithm presented in this paper consists in the replacement of a Dirichlet problem and an integral condition for a stationary transport equation with the same problem for an equation of the form:

\[ Lu = f \] (1)

We prove that the operator \( L \) is positive definite, hence there is only one solution for this new problem. An exact solution for equation (1) was found only in particular cases with the methods of mathematical analysis and functional analysis [1], [8], [14]. Numerical solutions were obtained with the help of following methods: Fourier transform, Laplace transform, the least squares, the finite element, Monte Carlo, truncated series of Chebyshev polynomials, [3]-[13].

In literature, we mention the existence of numerical solutions for the hyperbolic equations accompanied by the initial, boundary and integral conditions, [2]. Numerical examples that are presented in this paper prove the accuracy and computational efficiency of the proposed method.

2 Problem formulation

Let us consider the integral-differential equation of transport theory for the stationary case:

\[ \frac{\partial u(x,\mu)}{\partial x} + u(x,\mu) = \int_{-1}^{1} u(x,\mu') d\mu' + g(x,\mu) \] (2)

\[(x, \mu) \in D, \quad D = [0,1] \times [-1,1] = D_1 \times D_2\]

with

\[ u(0, \mu) = u(1, \mu) = 0, \]

\[ \int_{0}^{1} u(x, \mu) dx = s(\mu), \quad \forall \mu \in D_2 \] (3)

where \( g \) is the source function and \( u \) is the density of neutrons. These migrate towards a direction defined by the angle \( \alpha \) against \( Ox \) axis and we denote \( \mu = \cos \alpha \). Let us consider that the functions, which appear in (2) are continuous together with their first and second derivatives in the domain \( D \).
We define now a partition $\Delta_1= (x_0, x_1, \ldots, x_n)$ of the interval $D_1$ into $n$ subintervals of length $h$. Also, let $\Delta_2 = (\mu_0, \mu_1, \ldots, \mu_p)$ be a partition of the interval $D_2$, and $\tau = \mu_{i+1} - \mu_i$, $\forall i = 0, 1, \ldots, p - 1$. In order to solve the problem (2) - (3), we replace this problem with the partial derivative with respect to variable $x$ of (2):

$$\frac{\partial^2 u(x, \mu)}{\partial x^2} + \frac{\partial u(x, \mu)}{\partial x} = \int_{-1}^{1} \frac{\partial u(x, \mu')}{\partial x} d\mu' + \frac{\partial g(x, \mu)}{\partial x}$$

and using again (2) we obtain

$$\frac{\partial^2 u(x, \mu)}{\partial x^2} = -u = \gamma(x, \mu) \quad (4)$$

where

$$\gamma(x, \mu) = \int_{-1}^{1} g(x, \mu') d\mu' + \frac{\partial g(x, \mu)}{\partial x} - g(x, \mu) \quad (5)$$

For every value $\mu_i \in D_2$, $u \in L_2(D_1)$, the Hilbert space with the scalar product defined by the formula:

$$<w, v> = \int_0^1 w(x)v(x)dx \quad (6)$$

and the norm of $w$ is

$$\|w\| = \sqrt{<w, w>}$$

Now, we write (4) as an operator equation

$$Lu = f \quad (7)$$

where

$$Lu = -\frac{d^2 u(x, \mu_i)}{dx^2} + u(x, \mu_i)$$

$$f(x, \mu_i) = -\gamma(x, \mu_i)$$

$$u(0, \mu_i) = u(1, \mu_i) = 0 \text{ and } \int_0^1 u(x, \mu_i)dx = s(\mu_i) \quad (8)$$

Let us now suppose that the functions $u$, $g$ and $r$ belong to the Hilbert space $L_2(D)$, where $D = D_1 \times \mathbb{R}^n$, and we will prove that $L$ is a positive definite operator. Then (7) will have only one solution, [13].

First, we show that $L$ is a positive operator, namely

$$(Lu, u) \geq 0 \quad (9)$$

According to (6), we obtain

$$(Lu, u) = \int_0^1 \left(-\frac{d^2 u}{dx^2} + u\right) u dx = \int_0^1 \left(-u \frac{d^2 u}{dx^2} + u^2\right) dx =$$

$$= \int_0^1 \left(\frac{du}{dx}\right)^2 + u^2 \right) dx \geq 0 \quad (10)$$

On the other hand, an operator is called positive definite, if the following inequality is satisfied

$$(Lu, u) \geq C^2 \|u\| \quad (11)$$

In view of Cauchy-Schwarz inequality, we have

$$u^2(x, \mu_i) = \left(\int_0^1 \frac{1}{2} u'(t, \mu_i)\right)^2 \leq \int_0^1 \left(\frac{du}{dt}\right)^2 dt =$$

$$= \int_0^1 (u')^2 dt \leq \int_0^1 \left(\frac{du}{dt}\right)^2 + u^2 dt \quad (12)$$

Integrating both sides of (12) with respect to $x$ between 0 and 1, we get the inequality

$$\int_0^1 u^2(x, \mu_i)dx \leq \int_0^1 \left(\frac{du}{dt}\right)^2 + u^2 dt \quad (13)$$

It can be written symbolically as

$$\|u\|^2 \leq (Lu, u) \quad (13)$$

Therefore (11) is proved, the constant $C$ being equal here to unity.

3 Spectral method

Problem (7) - (8) is a nonstandard boundary problem. The presence of the integral condition can complicate the application of spectral methods and boundary elements methods. In our algorithm, we replace (7) - (8) with an equivalent problem, using the superposition principle.

Let us consider a new unknown function

$$u(x, \mu) = v(x, \mu) + w(x, \mu) \quad (14)$$

where
\[ w(x, \mu) = C_1(\mu)x + C_2(\mu) \] and
\[ \int_0^1 w(x, \mu)dx = s(\mu) \]

Therefore
\[ \int_0^1 v(x, \mu)dx = 0. \tag{15} \]

Functions \(C_1\) and \(C_2\) of \(\mu\) will be determined by the Dirichlet and integral conditions (8) and we get
\[ C_1(\mu) = v(0, \mu) - v(1, \mu) \]
\[ C_2(\mu) = -v(0, \mu) \tag{16} \]

where
\[ C_1(\mu) + 2C_2(\mu) = 2s(\mu) \tag{17} \]

The problem (7) – (8) becomes (18 – (20):
\[ \frac{\partial^2 v(x, \mu)}{\partial x^2} - v(x, \mu) - (v(0, \mu) - v(1, \mu))x + v(0, \mu) = \gamma(x, \mu) \tag{18} \]

where \(\gamma(x, \mu)\) is defined in (5);

- boundary condition
\[ v(0, \mu) + v(1, \mu) = -2s(\mu) \tag{19} \]

- and nonlocal condition
\[ \int_0^1 v(x, \mu)dx = 0 \tag{20} \]

Let us now consider the function \(V(x) = v(x, \mu_0)\) for a fixed \(\mu_0 \in [0, 1]\), that belongs to the above partition \(\Delta_2\).

The function \(V(x)\) is square integrable in \([0, 1]\). Using a spectral method, we can write \(V(x)\) in the form
\[ V(x) = \sum_{k=0}^n a_k(\mu_0)\omega_k(x) = \sum_{k=0}^n \alpha_k \omega_k(x) \tag{21} \]

where \(\omega_0(x)\) are the orthogonal functions and \(a_k = a_k(\mu_0)\), for \(\mu_0\) fixed. In practice, it is used for \(V\) the approximation
\[ v(x, \mu) = V(x) \approx \sum_{k=0}^n \alpha_k \omega_k(x) \tag{22} \]

The coefficients \(\alpha_0\) are chosen, such that to minimize the mean integral square error

\[ e = \int_0^1 \left(V(x) - \sum_{k=0}^n \alpha_k \omega_k(x)\right)^2 dx \tag{23} \]

and are given by
\[ \alpha_k = \int_0^1 V(x)\omega_k(x)dx \tag{24} \]

Next, we take
\[ \omega_k(x) = \cos k\pi x \tag{25} \]

and the solution of our problem (9) – (10)
\[ V(x) = \sum_{k=0}^m \alpha_k \cos k\pi x \tag{26} \]

Using the integral condition (20), we get \(\alpha_0 = 0\).

The solution (26) for \(n = 2m, m \in \mathbb{N}\), is introduced in equation (18) and using the spectral – Galerkin method, we get
\[ \sum_{k=0}^{2m} \left[ -\pi^2 k^2 \cos(k\pi x) - \alpha_k \cos(k\pi x) - 2(\sum_{k=1}^{2m} \alpha_{2k-1})x - s(\mu_0) + \sum_{k=1}^{2m} \alpha_{2k-1}\right], \cos j\pi x > \]
\[ = \langle \gamma(x, \mu), \cos j\pi x \rangle, \forall j \in \{1, 2, \ldots, 2m\} \tag{27} \]

where
\[ \int_0^1 \cos(k\pi x) \cdot \cos(j\pi x)dx = \begin{cases} 0, & k \neq j \\ 1/2, & k = j \end{cases} \]
\[ \int_0^1 x \cos(j\pi x)dx = -2 \frac{2}{(2l-1)^2 \pi^2}, \text{ for } j = 2l - 1, \]
\[ l = 1, 2, \ldots, m \]

Because \(\{\cos(k\pi x)\}_{k=0}^{2m}\) are orthogonal functions on \([0, 1]\), the system (27) becomes
\[ A\alpha = F \tag{28} \]

where \(A\) is of the form
\[ \begin{bmatrix} -\pi^2 - 1 + \frac{4}{\pi^2} & 0 & \frac{4}{\pi^2} & 0 & \ldots & 0 \\ 0 & -4\pi^2 - 1 & 0 & 0 & \ldots & 0 \\ \frac{4}{3^2\pi^2} & 0 & -9\pi^2 - 1 + \frac{4}{9\pi^2} & 0 & \ldots & 0 \\ 0 & 0 & 0 & -4^2\pi^2 - 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -n^2\pi^2 - 1 \end{bmatrix} \]

and

\[ F_j(\mu_l) = 2\int_0^1 \gamma(x, \mu_l) \cos \pi j x \, dx, \quad j \in \{1, 2, \ldots, n\} \]  

Solving the systems (28) for every \( l \in \{1, 2, \ldots, p - 1\} \), we can evaluate the solution values of our problem (18) - (20) in the nodes \( \mu_l \)

\[ v(x, \mu_l) = \sum_{k=0}^{\infty} a_k(\mu_l) \cos k\pi x, \quad l \in \{1, \ldots, p\} \]  

Finally, using (14) and (16) we obtain the solution \( u \) in any node \( \mu_l \)

\[ u(x, \mu_l) = v(x, \mu_l) + (v(0, \mu_l) - v(1, \mu_l)) \cdot x - v(0, \mu_l) \]  

4 Numerical examples

I. Let us consider the problem

\[ \frac{\partial u(x, \mu)}{\partial x} + u(x, \mu) = \int_{-1}^{1} u(x, \mu') d\mu' + g(x, \mu) \]

\[ g(x, \mu) = \mu \cos(2\pi x) - 2\pi \mu \sin(2\pi x) - \mu \]  

\((x, \mu) \in D, \quad D = [0, 1] \times [-1, 1] = D_1 \times D_2 \)

with \( u(0, \mu) = u(1, \mu) = 0 \),  

\[ \int_0^1 u(x, \mu) dx = -\mu, \quad \forall \mu \in D_2 \]  

Let us consider \( n = 4 \) in (30) and \( m = 2 \) in (27). In matrix form, for each \( \mu_l \in \{0.25, 0.5, 0.75\} \), we obtain a matrix equation (28), where

\[ A = \begin{bmatrix} -\pi^2 - 1 + \frac{4}{\pi^2} & 0 & \frac{4}{\pi^2} & 0 & \ldots & 0 \\ 0 & -4\pi^2 - 1 & 0 & 0 & \ldots & 0 \\ \frac{4}{3^2\pi^2} & 0 & -9\pi^2 - 1 + \frac{4}{9\pi^2} & 0 & \ldots & 0 \\ 0 & 0 & 0 & -4^2\pi^2 - 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -n^2\pi^2 - 1 \end{bmatrix} \]

and

\[ \gamma(x, \mu) = -\mu\left(4\pi^2 + 1\right)\cos(2\pi x) - 1 \]  

Solutions of the equation (20) are compared with the exact solution \( u_{\text{ex}} = \mu \cos(2\pi x) - \mu \)

The table 1 demonstrates that presented algorithm leads to an exact solution for our problem. We mention that the partition of the interval \([0, 1]\) is in these numerical cases

\[ \Delta_1 = (0; 0.25; 0.5; 0.75; 1) \]

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
</table>

\[ u_{\text{ex}} = 2\mu \cos(2\pi x) + \mu \cos(3\pi x) + 2\mu x - 3\mu \]  

II. In the second numerical example the function \( g \) is of the form

\[ g(x, y) = 2\mu[\cos(2\pi x) - 2\pi \sin(2\pi x)] + \mu[\cos(3\pi x) - 3\pi \sin(3\pi x)] + 2\mu x - \mu \]

and

\[ s(\mu) = -2\mu \]

The exact solution is now of the form

\[ u_{\text{ex}} = 2\mu \cos(2\pi x) + \mu \cos(3\pi x) + 2\mu x - 3\mu \]
Using the same values for \( n, m \) and \( p \), we obtain the following results for \( u \)

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( u )</th>
<th>( u_{\text{ex}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>( x_1 ) = -0.809</td>
<td>( x_1 ) = -0.802</td>
</tr>
<tr>
<td></td>
<td>( x_2 ) = -1</td>
<td>( x_2 ) = -1</td>
</tr>
<tr>
<td></td>
<td>( x_3 ) = -0.191</td>
<td>( x_3 ) = -0.198</td>
</tr>
<tr>
<td>0.5</td>
<td>( x_1 ) = -1.617</td>
<td>( x_1 ) = -1.604</td>
</tr>
<tr>
<td></td>
<td>( x_2 ) = -2</td>
<td>( x_2 ) = -2</td>
</tr>
<tr>
<td></td>
<td>( x_3 ) = -0.383</td>
<td>( x_3 ) = -0.396</td>
</tr>
<tr>
<td>0.75</td>
<td>( x_1 ) = -2.426</td>
<td>( x_1 ) = -2.405</td>
</tr>
<tr>
<td></td>
<td>( x_2 ) = -3</td>
<td>( x_2 ) = -3</td>
</tr>
<tr>
<td></td>
<td>( x_3 ) = -0.574</td>
<td>( x_3 ) = -0.595</td>
</tr>
</tbody>
</table>

The Fig. 1 shows the variations of solution \( u \) : \( u_1 (\mu = 0.25) \) - red color; \( u_2 (\mu = 0.5) \) – blue color; \( u_3 (\mu = 0.75) \) - green color and exact solution \( u_{\text{ex}} \) : \( u_{\text{ex}1} (\mu = 0.25) \); \( u_{\text{ex}2} (\mu = 0.5) \); \( u_{\text{ex}3} (\mu = 0.75) \).

We observe that the values in modulus of \( u \) increase with increasing of \( \mu \).

Error analysis leads to the conclusion that their values are lower than the value 0.021.

The surface that correspond to \( u(x_i, \mu_l) \) (denoted by \( U_1 \)), \( i \in \{0, 1, \ldots, 4\}, l \in \{0, 1, \ldots, 4\} \) is shown in the Fig. 2.

### 5 Conclusions

This paper contains the following original results on the study of integro-differential equations:

1. A method that replaces a boundary problem for one dimensional transport equation accompanied by an integral condition with a boundary problem and a nonlocal condition for a partial differential equation of second order.
2. Demonstration regarding the positivity of the operator \( L \) from equation (7).
3. Algorithm and spectral method for solving the new problem (7)-(8).

The results presented in Table 1 and Table 2 show that even using only four terms in (22) and five nodes into \( \Delta_1 \) and \( \Delta_2 \), the solutions obtained with our algorithm have acceptable values.

The existence of non-local condition in solving the transport equation will lead to the development of new directions of research on such phenomena.

This algorithm will be extended in a future paper to a boundary value problem with an integral condition for a transport equation with variable coefficients.

### References


