Mathematical Modeling and Analysis of a Vehicle Crash

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Abstract: Because of the fact that vehicle crash tests are complex and complicated experiments it is advisable to establish their mathematical models. This paper contains an overview of the kinematic and dynamic relationships of a vehicle in a collision. There is also presented basic mathematical model representing a collision together with its analysis. The main part of this paper is devoted to methods of establishing parameters of the vehicle crash model and to real crash data investigation i.e. – creation of a Kelvin model for a real experiment, its analysis and validation. After model's parameters extraction a quick assessment of an occupant crash severity is done.

Key-Words: Modeling, vehicle crash, Kelvin model, data processing.

1 Introduction

The main objective of this project is to establish a mathematical model of a vehicle collision. The purpose of this task is to simulate how the crash looks like – i.e. what are the main parameters describing the collision – without performing any real test. Real world experiments are difficult to realize – there are needed appropriate facilities, measuring devices, data acquisition process, qualified staff and of course – a car. Therefore it is justified to propose a mathematical model of a collision and analyze it instead of a real experiment to approximate its results.

In our main interest it is to analyze in details a Kelvin model. Having knowledge concerning one such a system we are able to extend the model e.g. to a couple of Kelvin elements in order to obtain a more accurate response (we can represent car elements and connections between them exactly by multiple spring – mass – damper models). Many researches have been done so far in the area of vehicle crash modelling.

Yang et al. [1] presented a feasibility study of using numerical optimization methods to design structural components for crash. The presented procedure required several software, which included parametric modeling (Pro/ENGINEER), automatic mesh generation (PDA PATRAN3), nonlinear finite element analysis (RADIOSS), and optimization programs. It was found that crash optimization was feasible but costly and that finite element mesh quality was essential for successful crash analysis and optimization.

Mahmood et al. [2] have described in detail a procedure for rapid simulation and design of the frame of an automotive structure. They developed a simplified program, called V-CRUSH, for rapid simulation of the structure. Correlation between the experimental and simulation results was very good.

Huang et al. [3] described Ford's Energy Management System that used CRUSH (Crash Reconstruction Using Static History) lumped mass modelling capability. Using the system, barrier loads and passenger compartment loads were calculated and compared to the test results in a frontal crash.

Above brief overview of the literature has been done according to Kim et al. [4].

In this paper we cover the spring – mass – damper modeling of the vehicle crash. We start with an overview of Kelvin model – an element in which mass is attached to spring and damper which are connected in parallel. Subsequently we give information about factors which determine crash severity for an occupant during collision. The largest part of this work is devoted to answer the following question – how to establish a model from real crash data? After presenting two methods for solution of this problem we proceed to analysis measurements from real collision.

2 Vehicle collision simulation – Kelvin model

A Kelvin model is shown in Fig. 2.1. It contains a mass together with spring and damper connected in parallel. This model can be utilized to simulate the vehicle-to-vehicle (VTV) collision, vehicle-to-barrier collision (VTB) as well as for component impact modeling. In majority of cases the response of the system is underdamped therefore we focus on this type of behavior.

ISSN: 1790-5117 ISBN: 978-960-474-178-6

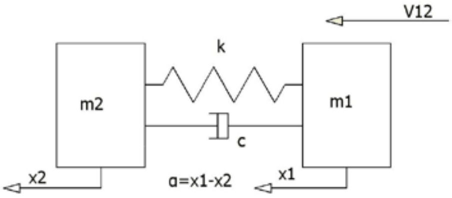

Fig. 2.1: Kelvin model

2.1 Underdamped system (1>ζ>0)

Equation of motion (EOM):

$$\ddot{\alpha} + 2\zeta\omega_e\dot{\alpha} + \omega_e^2\alpha = 0 \qquad (2.1)$$

where $\zeta = \dfrac{c}{2m\omega_e}$ and $\omega_e = \sqrt{\dfrac{k}{m}}$

Transient responses of the underdamped system are:

$$\alpha(t) = \frac{v_0 e^{-\zeta\omega_e t}}{\sqrt{1-\zeta^2}\,\omega_e}\sin(\sqrt{1-\zeta^2}\,\omega_e t) \qquad (2.2)$$

displacement (dynamic crush)

$$\dot{\alpha}(t) = v_0 e^{-\zeta\omega_e t}[\cos(\sqrt{1-\zeta^2}\,\omega_e t) - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\sqrt{1-\zeta^2}\,\omega_e t)] \qquad (2.3)$$

velocity

$$\ddot{\alpha}(t) = v_0\omega_e e^{-\zeta\omega_e t}[-2\zeta\cos(\sqrt{1-\zeta^2}\,\omega_e t) + \frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin(\sqrt{1-\zeta^2}\,\omega_e t)] \qquad (2.4)$$

deceleration

We see that above closed – form results are complex. To obtain the responses of the Kelvin model we use Matlab Simulink software.

In the analysis of the crash pulse (deceleration) alongside with velocity and displacement graphs we are able to observe specific relationships between them and between two timings: t_m – time of dynamic crush and t_f – time of rebound (or time of separation velocity). Those dependences are shown in Fig. 2.2. The values on the graph below are just for presenting the principle – they do not come from any experiment.

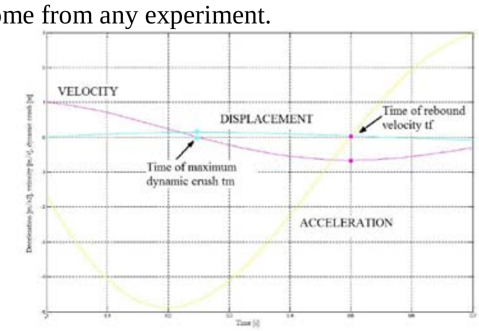

Fig. 2.2: Relationships between t_m, t_f and acceleration, velocity, displacement

At t_m the corresponding velocity is zero and the dynamic crush reaches its maximum value. At t_f the corresponding deceleration is zero and velocity reaches its maximum value. Please note that t_f is twice as long as t_m (in Fig. 2.2 t_f=0.5s and t_m=0.25s).

2.2 Coefficient of restitution (COR)

In the impact of the dynamic system the coefficient of restitution (COR) is defined as the ratio of relative separation velocity to the relative approach velocity. During the deformation phase, the relative approach velocity decreases from its initial value to zero due to the action of the deformation impulse, as shown in Fig. 2.3.

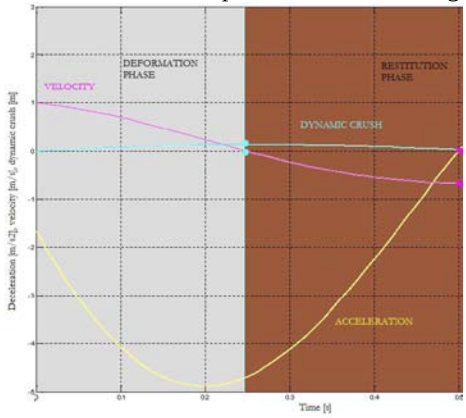

Fig. 2.3: Deformation and restitution phase during a crash

At the time when the relative approach velocity is zero, the maximum dynamic crush occurs. The relative velocity in the rebound phase then increases negatively up to the final separation (or rebound) velocity, at which time the two masses separate from each other (or a vehicle rebounds from the barrier). At the separation time, there is no more restitution impulse acting on the masses, therefore, the relative acceleration at the separation time is zero [5]. To derive the relationship between the coefficient of restitution and damping factor of the system we use (2.4).

At the time of separation ($t=t_r=t_f$) the relative deceleration $\ddot{\alpha} = 0$

Therefore from (2.4):

$$2\zeta\cos(\sqrt{1-\zeta^2}\,\omega_e t) + \frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin(\sqrt{1-\zeta^2}\,\omega_e t) = 0$$

We rewrite it in the following form:

$$\tan(\sqrt{1-\zeta^2}\,\omega_e t) = \frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1}$$

from Pythagorean Theorem we get:

$$\cos(\sqrt{1-\zeta^2}\,\omega_e t) = 2\zeta^2 - 1 \qquad (2.5)$$

COR=relative separation velocity/relative approach velocity = $\dfrac{\dot{\alpha}(t)}{v_0}$

$$COR = e^{-\zeta\omega_e t}[\cos(\sqrt{1-\zeta^2}\,\omega_e t) - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\sqrt{1-\zeta^2}\,\omega_e t)] \qquad (2.6)$$

There are three special cases:

1. No damping in the system ζ=0, then COR = 1.
2. Critically damped system ζ=1, then COR = 0.135.
3. Highly overdamped system ζ=∞, then COR = 0.

ISSN: 1790-5117

ISBN: 978-960-474-178-6

We can simplify (2.6) by substituting $\sin(\sqrt{1-\zeta^2}\,\omega_e t) = 2\zeta\sqrt{1-\zeta^2}$ so that we get:

$$COR = e^{-\left[\frac{\zeta}{\sqrt{1-\zeta^2}}\arccos(2\zeta^2-1)\right]} \qquad (2.7)$$

3 Bases of occupant – vehicle modeling

In this section we present basic notions and terms needed to asses the crash severity for an occupant. As the crash pulse approximation we use an ESW (Equivalent Square Wave). Fig. 3.1 shows an unbelted occupant in a vehicle during a collision.

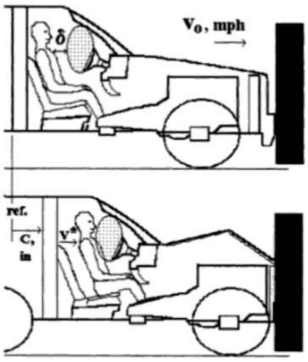

Fig. 3.1: Occupant during collision [5]

v_0 – initial vehicle rigid barrier impact velocity
v^* – occupant to interior surface contact velocity
δ – occupant free travel space (restraint slack)
c – vehicle dynamic crush at time t
t^* – time when occupant contacts restraint
t_m – time of dynamic crush

EOM for vehicle:

$$\ddot{x}_v = -\frac{F}{M} = -ESW \qquad (3.1)$$

$$\dot{x}_v = v_0 - ESWt \qquad (3.2)$$

$$x_v = v_0 t - \frac{1}{2}ESWt^2 \qquad (3.3)$$

EOM for occupant:

$$\ddot{x}_o = -ESW[1-\cos(p) + \omega t^* \sin(p)] \qquad (3.4)$$

$$\dot{x}_o = \dot{x}_v + \frac{ESW}{\omega}[\sin(p) + \omega t^* \cos(p)] \qquad (3.5)$$

$$x_o = x_v + \delta + \frac{ESW}{\omega^2}[1-\cos(p) + \omega t^* \sin(p)] \qquad (3.6)$$

$$\ddot{x}_o|_{\max} = -ESW[1+\sqrt{1+(\omega t^*)^2}\,] \qquad (3.7)$$

where $p = \omega(t - t*)$ for $t \geq t*$ and $\omega = \sqrt{\frac{k}{m_{occupant}}}$

$$t* = \sqrt{\frac{2\delta}{ESW}} \qquad (3.8)$$

restraint contact time

3.1 Prediction of occupant deceleration using DAF

Let us define dynamic amplification factor as the ratio of maximum occupant chest deceleration to the ESW:

$$DAF = \frac{\ddot{x}_o|_{\max}}{ESW} = \frac{-ESW[1+\sqrt{1+(\omega t^*)^2}\,]}{ESW}$$

$$DAF = 1+\sqrt{1+(\omega t^*)^2} = 1+\sqrt{1+(2\pi f t^*)^2} \qquad (3.9)$$

where $\omega = 2\pi f$ and f is restraint natural frequency.

Since $DAF = \gamma = 1+\sqrt{1+(2\pi f t^*)^2}$ and we approximate the crash pulse by ESW we can write that the maximum occupant chest deceleration is given by:

$a_0 = ESW \cdot DAF = ESW[1+\sqrt{1+(2\pi f t^*)^2}\,]$ where $v^* = ESW \cdot t^*$

therefore

$$a_0 = ESW + \sqrt{ESW^2 + (2\pi f v^*)^2} \qquad (3.10)$$

It is a common practice to install in trucks pretensioners. This is because of the fact that the ESW of a truck is higher that that of a car. Therefore if we want to decrease the occupant deceleration we need to decrease the restraint slack and that is justified by the DAF relationship.

4 Obtaining parameters of the Kelvin model from tests

Fig. 4.1 presents a Kelvin model of a vehicle-to-barrier impact.

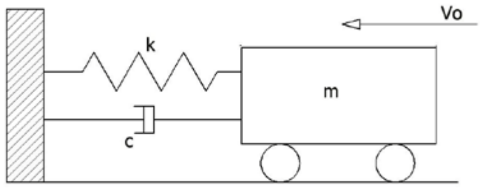

Fig. 4.1: VTB collision – Kelvin model

k – spring stiffnes
c – damping coefficient
m – mass of the vehicle
v_0 – barrier initial impact velocity

4.1 Method 1 - analytical

To obtain structural parameters k and c first we need to determine two other parameters: ζ – damping factor and f – structure natural frequency. Before we do that let us first remind the centroid time concept.

Centroid time – it is a time at the geometric center of area of the crash pulse from time zero to the time of dynamic crush. We define it as follows:

$$t_c = \frac{C}{v_0} \qquad (4.1)$$

ISSN: 1790-5117 ISBN: 978-960-474-178-6

We define normalized centroid time and angular position at dynamic crush as:

$$\tau_c = t_c\omega_e = \left(\frac{\alpha_m}{v_0}\right)\omega_e = e^{-\zeta\tau_m}$$

$$\tau_m = t_m\omega_e = \frac{1}{\sqrt{1-\zeta^2}}\arctan\frac{\sqrt{1-\zeta^2}}{\zeta}$$

where α_m is the maximum dynamic crush.

After transforming above two equations we get relative centroid location:

$$\frac{\tau_c}{\tau_m} = \frac{t_c}{t_m} = \frac{\sqrt{1-\zeta^2}}{\arctan\frac{\sqrt{1-\zeta^2}}{\zeta}} e^{\left[\frac{-\zeta}{\sqrt{1-\zeta^2}}\arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right]} \quad (4.2)$$

$\tau_m = t_m(2\pi f)$ so

$$ft_m = \frac{1}{2\pi\sqrt{1-\zeta^2}}\arctan\frac{\sqrt{1-\zeta^2}}{\zeta} \quad (4.3)$$

Once we find the relative centroid location by determining t_c and t_m we can get damping factor ζ from (4.2).
After deriving damping factor ζ and knowing time of dynamic crush t_m we obtain the value of structure natural frequency from (4.3).

Having already values of ζ and f we determine structural parameters of the model – k and c:

Since $\omega_e = 2\pi f = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2m\omega_e}$

$$k = 4\pi^2 f^2 m \quad (4.4)$$

$$c = 4\pi f\zeta m \quad (4.5)$$

In order to estimate the parameters of the Kelvin model basing on the real crash pulse data we just need main information concerning the collision: time of dynamic crush t_m, initial impact velocity v_0, dynamic crush C and mass of the vehicle m. Taking into consideration the complexity of the collision phenomena it is a significant advantage – we can e.g. asses the stiffness and damping of a frontal structure of a car using simple data mentioned above.

4.2 Method 2 – Using Matlab Identification Toolbox

This Toolbox allows us to obtain the parameters of the system according to the input and output data. As an example we are going to use the Simulink model of the second order differential equation (second order oscillating element). The forcing factor is the external force over mass (acceleration) – initial conditions (velocity and displacement) are set to zero.

Data:
F = 300N; k = 100N/m; c = 5N-s/m; m = 3kg; $v_0 = d_0 = 0$

Equation of second order oscillating element is [6]

$$T^2\frac{d^2y(t)}{dt^2} + 2\zeta T\frac{dy(t)}{dt} + y(t) = Kx(t) \quad (4.6)$$

where y(t) – output and x(t) – input.

By taking Laplace transform of (4.6) with zero initial conditions we get:

$$T^2s^2Y(s) + 2\zeta TsY(s) + Y(s) = KX(s) \quad (4.7)$$

Therefore the transfer function of the system given by (4.7) is:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{K}{T^2s^2 + 2\zeta Ts + 1} \quad (4.8)$$

From the EOM of the Kelvin model we have:

$$mu(t) = m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky \quad (4.9)$$

input u(t) is an acceleration.

By taking Laplace transform of (4.9) with zero initial conditions we obtain the following transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{m}{ms^2 + cs + k} \quad (4.10)$$

(4.8) and (4.10) are describing the same model. Therefore they are equal to each other if and only if:

$T = \sqrt{\frac{m}{k}}$ and $K = \frac{m}{k}$ and $\zeta = \frac{c}{2m\omega_e}$ and $\omega_e = \sqrt{\frac{k}{m}}$

With this knowledge we proceed to Identification Toolbox. We select the appropriate type of estimation – in our case – since we use Kelvin model - an underdamped system with two poles.
Parameters obtained from estimation are shown in Fig. 4.4.

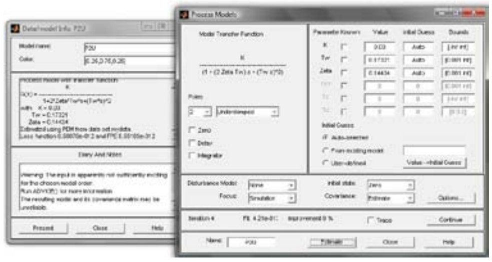

Fig. 4.4: Identification Toolbox - results

After obtaining the values which are describing the estimated model we check what are the values of T, K and ζ for our reference model – and we compare them with those ones from the estimated model.

For k = 100 N/m, c = 5N-s/m, m = 3kg we have:

$K = \frac{m}{k} = \frac{3}{100} = 0.03m/N$ ID toolbox: K = 0.03m/N

$T = \sqrt{\frac{m}{k}} = \sqrt{\frac{3}{100}} = 0.17321s$ ID toolbox: T = 0.17321s

$\zeta = \frac{c}{2m\omega_e} = \frac{5}{2\cdot 3\cdot\sqrt{\frac{100}{3}}} = 0.14434$ ID toolbox: $\zeta = 0.14434$

The results of approximation are perfect. Time constant T, damping coefficient ζ and gain K for both models –

reference and our estimated – are the same. It means that we can use Identification Toolbox to precisely determine what are the coefficients of the Kelvin model when we are given an input and an output of the system and the initial conditions are set to zero.

5 Investigation of real crash data

Let us now analyze data from the experiment.

5.1 Experiment procedure [7]

In the experiment conducted by UiA [7] the test vehicle, a standard Ford Fiesta 1.1L 1987 model was subjected to a central impact with a vertical, rigid cylinder at the initial impact velocity v_0 = 35km/h. Mass of the vehicle (together with the measuring equipment and dummy) was 873kg. Scheme of the experiment is shown in Fig. 5.1.

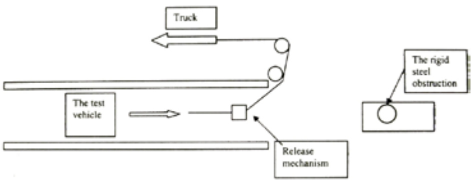

Fig. 5.1: Scheme of the test collision [7]

Vehicle accelerations in three directions (longitudinal, lateral and vertical) together with the yaw rate at the center of gravity were measured. Using normal-speed and high-speed video cameras, the behavior of the obstruction and the test vehicle during the collision was recorded.

5.2 Data processing

Since we are given the accelerations in 3 directions (longitudinal – x, lateral – y, vertical – z) we are able to propose 3 different Kelvin models for every direction. Because of the fact that we are mostly interested in what happens in the direction in which a car hits the obstacle, we are going to analyze x – direction (longitudinal).
To approximate the crash pulse we use Curve Fitting Toolbox with Gaussian approximation as it is shown in Fig. 5.2.

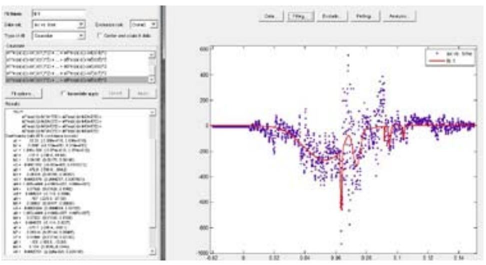

Fig. 5.2: Curve Fitting Toolbox – preparation of measured data

To obtain the velocity curve we integrate the approximated pulse – the result is shown in Fig. 5.3.

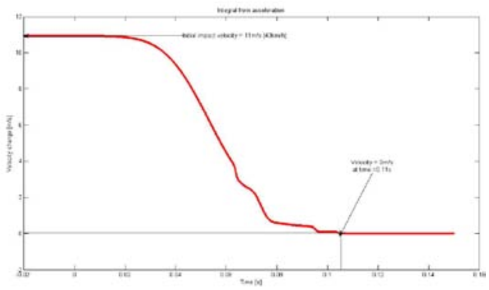

Fig. 5.3: Velocity obtained from measured acceleration

We see in Fig. 5.3 that the initial velocity is not equal to 35km/h as it was stated in the experiment's description but is 5km/h higher. This discrepancy is a result of using raw data – without filtering. From this plot we read the value of time of dynamic crush t_m = 0.11s.
To get the displacement graph we proceed in the manner described above – we approximate and integrate the velocity curve from Fig. 5.3. The plot of displacement is shown in Fig. 5.4.

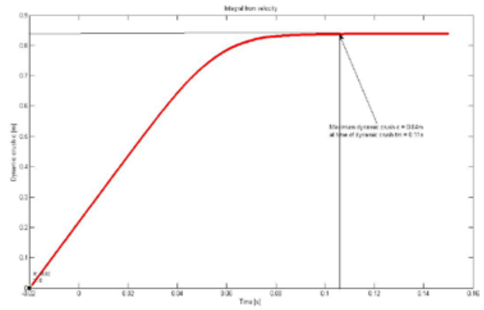

Fig. 5.4: Displacement obtained from measured acceleration

From the plot we determine maximum dynamic crush C = 0.84m at time of dynamic crush = 0.11s.

5.3 Comparison between model and real data according to method 1

Knowing values of v_0 = 11m/s, t_m = 0.11s, C = 0.84m and m = 873kg from the real test, using method described in Section 4.1 we determine parameters: t_c, t_c/t_m, ζ, f, k, c:

$$t_c = \frac{C}{v_0} = \frac{0.84m}{11m/s} = 0.076s$$

$$\frac{t_c}{t_m} = \frac{0.076s}{0.11s} = 0.69$$ and furthermore from (4.2):

ζ = 0.05 and from (4.3): ft_m = 0.24Hz-s

$$ft_m = 0.24Hz - s \quad \text{so} \quad f = \frac{0.24Hz - s}{t_m} = \frac{0.24Hz - s}{0.11s} = 2.2Hz$$

We calculate the parameters of the Kelvin model:

$$k = 4\pi^2 f^2 m = 4\pi^2 (2.2Hz)^2 \cdot 873kg = 166809 N/m$$

spring stiffness

$$c = 4\pi f \zeta m = 4\pi \cdot 2.2Hz \cdot 0.05 \cdot 873kg = 1207 N - s/m$$

damping coefficient

Having parameters of the Kelvin model we investigate its response using the Simulink diagram with the initial velocity v_0 = 11m/s. The result is shown in Fig. 5.5.

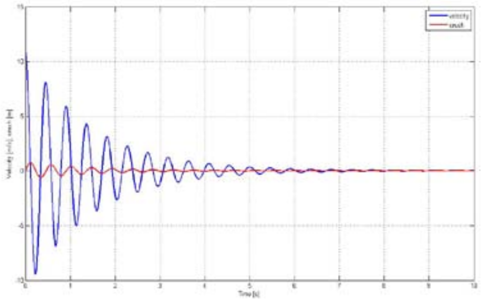

Fig. 5.5: Velocity and displacement vs time of the Kelvin model with estimated parameters

That is the response of the mass for 10 seconds. It is a typical one for the second order oscillating element – also Kelvin model.

In Fig. 5.6 you see the response in time interval used in the test data analysis (a magnified part of above plot).

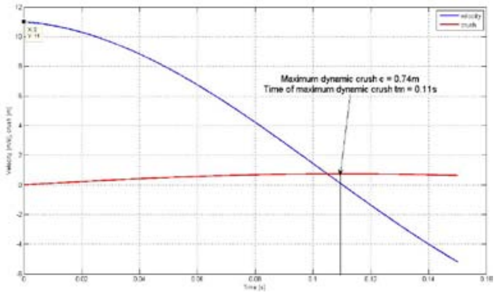

Fig. 5.6: Final analysis – velocity and displacement vs time of the Kelvin model with estimated parameters in the crash interval

Although the approximation of the velocity curve is not quite exact – we do not see e.g. a rebound, still the accuracy of approximation is very good. Time of dynamic crush t_m obtained from the model is exactly the same as in experiment: t_m = 0.11s and maximum dynamic crush C = 0.74m is about 12% less than that from the real test.

5.4 Estimation of maximum chest deceleration of occupant

Knowing initial impact velocity v_0 = 11m/s, maximum dynamic crush C = 0.84m, time when it occurs t_m = 0.11s and distance between an occupant and vehicle (restraint slack) δ = 0.6m we calculate:

$$ESW = 0.5\frac{v_0^2}{C} = 0.5\frac{11^2}{0.84} = 72m/s^2$$

$DAF = \frac{a_0}{ESW} = 1 + \sqrt{1 + (2\pi f t^*)^2}$, where $t* = \sqrt{\frac{\delta}{c}} t_m$ is the time when occupant contacts restraint, f is restraint natural frequency (we assume following [5] a typical value of f = 6Hz) and a_0 is the maximum occupant chest deceleration.

$$DAF = 1 + \sqrt{1 + (2\pi \cdot 6Hz \cdot \sqrt{\frac{0.6m}{0.84m}} 0.11s)^2} = 4.64$$

Maximum occupant chest deceleration:

$$a_0 = DAF \cdot ESW = 4.64 \cdot 72m/s^2 = 334m/s^2 = 34g$$

6 Conclusions

We have managed to prepare the crash data for analysis and extract the mathematical model from it. Challenges here were to choose an appropriate test data approximation and time interval in which we want to investigate the collision. Having this done we can determine maximum crush of a car, when it occurs, how the velocity changes and what are the changes in acceleration of a car during a crash. What is more – we have also estimated the maximum occupant deceleration – that is one of the main tasks in the area of crashworthiness study.

When it comes to the further work, we plan to extend our simple spring – mass – damper model to multiple Kelvin elements system. Then we will obtain more accurate results and – what is also important – for particular car components, not for a car as a one element. The other thing which could improve the results is using a Maxwell model (a mass together with a spring and damper connected in series) for a vehicle to rigid pole crash simulation. This system gives better approximation of offset impacts and localized pole collisions because it provides more accurate response for longer times of maximum dynamic crush. The last improvement is to filter the accelerometer measurements and to use more accurate type of curve approximation.

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