An Application of Fuzzy Hypotheses Testing in Radar Detection

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Abstract-A method of testing of fuzzy hypotheses for fuzzy data is applied to radar detection process. Imprecise parameters of the distribution under each hypotheses are modeled as a fuzzy number in which we consider that the received signal data is fuzzy instead we consider that the received signal data is crisp. The advantages in sorting fuzziness will be stated.

I. Introduction

Real observations of continuous quantities are not precise numbers, such observation are called fuzzy. The fuzziness is different from measurement probabilities, it is a feature of single observations from continuous quantities as in reference [1], probability theory is related to random phenomena, relative frequency and stochastic process; and, on the other hand, fuzzy set theory is related to non precise data, vague statements.

In section 2 we introduce the density function of both radar signal (target) and radar noise (no target).

Through section 3, we analyze and formulate the fuzzy test problem for fuzzy data as in reference [1].

In section 4, we will use the analysis and formulation that we introduce in section 3 to be applied for radar detection.

II. Radar Received Signal

There are two basic operations performed by radar are:

1. Detection of the presence of the reflecting objects.
2. Extraction of information from the received waveform to obtain the target data as position, velocity, angular position.

The detection and the extraction depends on each other because radar that is good detection device is usually a good radar for extracting information, and vice versa. But the problem is in detecting a signal in the presence of noise. Noise ultimately limits the capability of any radar.

We introduce here how to obtain the probability density function of radar signal and radar noise, so the joint probability density function of two random variable \( z(t) \) and \( \phi(t) \) according to reference [2] is given by:

\[
f(z, \phi) = \frac{z}{2\pi\sigma^2} \exp\left(-\frac{z^2 + \mu^2}{2\sigma^2}\right) \exp\left(-\frac{z\mu \cos\phi}{\sigma^2}\right)\tag{1}
\]

Where \( z(t) \) and \( \phi(t) \) represent the modulus and the phase of the received signal and \( \mu \) is the mean of the random variable \( z(t) \), \( \sigma \) represent the standard deviation.
The pdf for $z$ alone is obtained by integrating Eq.(1) over $\varphi$

$$f(z) = \int_0^{2\pi} f(z, \varphi)d\varphi = \frac{z}{\sigma^2} \exp\left(-\frac{z^2 + \mu^2}{2\sigma^2}\right) \times$$

$$\frac{1}{2\pi} \int_0^{2\pi} \exp\left(\frac{z \mu \cos \varphi}{\sigma^2}\right)d\varphi$$

(2)

Where the integral inside equation (2) is known as the modified Bessel function of zero order

$$I_0(\beta) = \frac{1}{2\pi} \int_0^{2\pi} e^{\beta \cos \theta} d\theta$$

(3)

$$f(z) = \int_0^{2\pi} f(z, \varphi) d\varphi = \frac{z}{\sigma^2} I_0\left(\frac{z \mu}{\sigma^2}\right) \exp\left(-\frac{z^2 + \mu^2}{2\sigma^2}\right)$$

(4)

Which is a rice probability density function. If $\frac{\mu}{\sigma^2} = 0$ (noise alone), Then Eq.(4) becomes the Rayleigh probability density function

$$f(r) = \frac{z}{\sigma^2} \exp\left(-\frac{z^2}{2\sigma^2}\right)$$

(5)

Also, when $\frac{\mu}{\sigma^2}$ is very large, Eq. (4) can be approximately a Gaussian probability density function of mean $\mu$ and variance $\sigma^2$:

$$f(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right)$$

(6)

### III. Testing of Fuzzy Hypotheses For Fuzzy Received signal

It has been noticed that almostly measurable quantities are imprecise by nature. For example, the mean power interference is not generally known but must be estimated by sampling. Even the power of a transmitted signal can be known only to certain degree of precision and may fluctuate slightly. We will model such imprecise quantities as fuzzy numbers.

In this section we analyze and formulate the fuzzy hypotheses test for fuzzy data (received signal) as introduced in reference [1], then we will make a comparison with fuzzy threshold and crisp threshold.

**The hypothesis model is given by:**

$H_o$: Interference (noise or no target)

$H_a$: Target + Interference (radar signal or it is a target)
Under hypothesis $H_i$, the probability density function $f(z_i; \theta_k)$, where $z_i$ is the input received signal, while $\theta_k$ is a vector of $\theta$ of $k$ imprecise parameters in which the probability density function depends.

Under hypothesis $H_o$, the probability density function is given by:

$$f(z_i; \theta_k; \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z_i - \mu)^2}{2\sigma^2}\right)$$

(7)

With mean zero and standard deviation $\sigma$.

Under hypothesis $H_A$, the probability density function is given by:

$$f(z_i; \theta_k; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z_i - \mu)^2}{2\sigma^2}\right)$$

(8)

With mean $\mu$ and standard deviation $\sigma$.

IV. Application

We will use the analysis and formulation in section 3 to make fuzzy hypotheses test for fuzzy data.

Given a set of $N$ observations $z_i = (z_1, ..., z_n)$ in which we want to test the hypothesis.

$H_o: \mu \leq 0$

$H_A: \mu > 0$

Now assume that the sample signals are symmetric triangular fuzzy numbers with support has an interval of size equal to 2 and 1 respectively as shown in figure below.

$$\mu = 1 \text{ and } \sigma^2 = 0.25.$$ In which $\mu = 1$ and $\sigma^2 = 0.25$ are symmetric triangular fuzzy with support has an interval of size equal to 2 and 1 respectively as shown in figure below.

$$\mu_{\mu, \sigma^2}(\mu), \mu_{\mu, \sigma^2}(\sigma^2)$$

Figure (2) Membership function of A symmetric (TFN) $\mu$ and $\sigma^2$.

Steps of analysis of this model:

1. Tabulate and Plot the density function $f(z_i)$ of the signal for the given values of $z_i$.

2. By using the extension principle plot $\eta(z_i)$, where $\eta(z_i)$ is the membership function of the test-statistic of normal distribution $T^*$, which is given by:

$$T^* = \left(\frac{\bar{z} - \mu}{\sigma / \sqrt{n}}\right)$$

(9)

Where,

$$\bar{z} = \frac{\sum_{i=1}^{n} z_i}{n}$$

(10)
\( \bar{z} \) will be calculated as discussed in reference[5], which is also a symmetric triangular fuzzy number with support equal to 0.2.

The membership function of the test-statistic is given by:

\[
\eta(z_i) = \sup \{ \min (\mu(\mu), \mu(\sigma^2)) \}
\]

3. Plot the characteristic function \( \eta(z_i) \) which is also a symmetric triangular fuzzy number over the density function \( f(z_i) \) of the normal distribution.

4. As shown in the figure we take \( \delta = 0.5 \) for example to calculate the \( \delta - \text{cut} \) of characterizing function \( \eta \) of the test statistic \( t^* \)

\[
C_{\delta}(0.5) = [t_1(0.5),t_2(0.5)] \quad (11)
\]
\[
C_{\delta}(0.5) = [0.6,1.6] \quad (12)
\]

5. Calculate the \( \delta - \text{cut} \) of \( p^* \) under hypotheses of right sided test

\[
C_{\delta}(p^*) = [p_1(0.5),p_2(0.5)] \quad (13)
\]
\[
C_{\delta}(p^*) = [P(T \geq t_2(0.5)),P(T \geq t_1(0.5))] \quad (14)
\]

The value of \( p_1(0.5) \) is the dark area as shown in figure (3)

\[
p_1(0.5) = P(T \geq t_2(0.5)) = P(T \geq 1.6) \quad (15)
\]
\[
p_1(0.5) = 1 - P(T < 1.47) = 1 - 0.9452 = 0.0548
\]

The value of \( p_2(0.5) \) is the light area + dark area as shown in figure (3)

\[
p_2(0.5) = P(T \geq t_1(0.5)) = P(T \geq 0.6) \quad (16)
\]
\[
p_2(0.5) = 1 - P(T < 0.6) = 1 - 0.7257 = 0.2743
\]

Then,

\[
C_{\delta}(p^*) = [0.0548,0.2743] \quad (17)
\]

6. Compare between the value of \( \alpha \) (type one error) which here is the probability of false alarm

\[
\alpha = P(\text{accept } H_0 / \text{H is true}) = P(\text{accept signal (target) / noise(no target) is true})
\]

So,

If \( \alpha = P_{fa} = 0.05 \)

The decision is accept \( H_0 \) (no target).

If \( \alpha = P_{fa} = 0.1 \)
The decision is neither accept nor reject $H_0$ (no target).
If $\alpha = P_{fa} = 0.25$

The decision is reject $H_0$ (no target).

Remark:
This can be done using matlab for calculation and will give.

Figure (4) Density $f(z_i)$ of standard normally distributed test statistic and characterizing function $\eta(z_i)$ of the test statistic $T^*$ by using matlab.

Calculation of crisp and fuzzy threshold

The crisp threshold is calculated according to reference [3].

$$v_T = \sqrt{2\sigma^2 \ln \left( \frac{1}{P_{fa}} \right)}$$

(18)

If $\alpha = P_{fa} = 1 \times 10^{-1}$ and $\mu = 1, \sigma^2 = 0.5$

$$v_T = \sqrt{2(0.5)^2 \ln \left( \frac{1}{1 \times 10^{-1}} \right)} = 1.517$$

(19)

So we can compare the fuzzy threshold with the crisp threshold for the following cases:

1. $\mu = 1$ and $\sigma^2 = 0.25$
2. $\mu = 1$ and $\sigma^2 = 0.5$
3. $\mu = 1$ and $\sigma^2 = 0.75$

Figure (5) Fuzzy threshold for various values of $\mu, \sigma^2$ with the Crisp threshold.

Calculation of membership function of likelihood ratio

The likelihood ratio function is defined as the ratio of the probability density functions under each hypothesis

$$\lambda(z_i; P_x, P_y) = \frac{f_{i, \lambda}(z_i, \theta)}{f_{i, \lambda}(z_i, \theta)} < v_T$$

(20)

According to the extension principle, the likelihood ration function can be written in terms of its membership function according to reference [1].

$$\mu_{\lambda(z_i)}(t) = \sup_{t = \lambda(z_i)} \left\{ \min_{\lambda \in \lambda(z_i)} \left[ \mu_{\lambda, \lambda}(\theta_{x, i}), ..., \mu_{\lambda, \lambda}(\theta_{x, k}) \right] \right\}$$

(21)

$$\lambda(z_i; \mu, \sigma^2) = \frac{z_i}{\sqrt{2\pi\sigma^2}} \exp \left( \frac{2\mu z_i - \mu^2}{2\sigma^2} \right)$$

(22)
To get membership function for the likelihood ratio we will use the extension principle.

\[ \mu_{\lambda(z_i)}(t) = \sup_{t = \lambda(z_i)} \{ \min \left[ \mu_{\lambda,\sigma^2}(\sigma^2), \mu_{A,\mu}(\mu), \mu_{A,\sigma^2}(\sigma^2) \right] \} \]  

(23)

Since;

\[ \mu_{\lambda,\sigma^2}(\sigma^2) = \mu_{A,\sigma^2}(\sigma^2) \]  

(24)

Then, equation (27) Becomes

\[ \mu_{\lambda(z_i)}(t) = \sup_{t = \lambda(z_i)} \{ \min \left[ \mu_{A,\mu}(\mu), \mu_{A,\sigma^2}(\sigma^2) \right] \} \]  

(25)

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<th>0.25</th>
<th>0.39</th>
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<td>1</td>
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<td>0.7</td>
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Figure (6) Membership function of the likelihood over the density function of the normal distribution.

The advantages for considering fuzzy data rather than crisp data:

1. We enhance the decision problem as here we have three region for decision criteria rather than two region of decision criteria.

2. The probability of detection increase as the threshold level in case we consider fuzziness to data became a function rather than a crisp value.

3. The uncertainty in the signal is considered due to imprecision but not due to randomness which is more realistic, as the probability theory is useful when we deal with random data.

V. Concluding Remark

We have demonstrated a method of signal detection based on fuzzy hypothesis testing for fuzzy received data, later we will drive:

1. The membership functions of probability of detection and probability of false alarm.

2. Test of hypotheses for different types of receiver like matched filter receiver, correlation receiver, the inverse probability receiver, delay line integrator, the binary integration receiver.

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References


