Fuzzy Covering Problem Based on the Expert Valuations

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Abstract: A new approach in the set covering problems is presented based on the expert knowledge presentations. An a priori uncertain information on the alternatives is given by some probability distribution and an a priori certain information on the knowledge competitions is given by some weights. A new criterion is introduced for a minimal fuzzy covering problem which is a minimal value of average misbelief in possible alternatives. A bicriterial problem is obtained using the new criterion and the criterion of covering average price minimization. The proposed approach is illustrated by an example.

Key–Words: Minimal fuzzy covering, Minimal compatibility level, Positive and negative discrimination measures, Average misbelief criterion, Bicriterial problem

1 Introduction

Optimization and decision-making problems are traditionally handled by either a deterministic or a probabilistic approach. The former provides an approximate solution, completely ignoring uncertainty, while the latter assumes that any uncertainty can be represented as a probability distribution. Of course, both approaches only partially capture reality uncertainty that indeed exists but not in the form of known probability distributions.

The existing literature clearly supports the notion of using the fuzzy set theory and soft-computing techniques to further expand the human capability in making optimal decisions involving non-probabilistic uncertainty [3], [9], [14]–[20], [23]–[33], [37]–[40]. In the Preface of the Journal of Fuzzy Optimization and Decision Making (vol. I, 2002, pp. 11–12) Professor L. A. Zadeh had said: “My 1970 paper with R.E. Bellman, ‘Decision-Making in a Fuzzy Environment’ was intended to suggest a framework based on the theory of fuzzy sets for dealing with imprecision and partial truth in optimization and decision analysis. In the intervening years, a voluminous literature on applications of fuzzy logic to decision analysis has come into existence.”

In particular, when constructing decision-making systems [3], [9], [14]–[18], [20]–[25], [27]–[33], [38]–[40], the use of fuzzy set theory is rather effective, since information fuzziness is a typical property of any system of this kind. Frequently, the main material for the construction of such systems is expert knowledge and representations. The use of such systems containing subjective, fuzzy uncertainty [11] leads to natural generalizations of the above-mentioned problems in the form of fuzzy optimization problems.

Fuzzy programming problems has been discussed widely in literature [3], [9], [14], [19], [20], [24], [27], [31], [33], [34] and applied in such various dis-
cielines as operations research, economic management, business administration, engineering and so on. Liu B. (Liu, [19] 2002) presents a brief review on fuzzy programming models, and classifies them into three broad classes: expected value models, chance-constrained programming and chance-dependent programming.

Our further study belongs to the first class, where we used the instrument of fuzzy statistics and fuzzy set theory for our investigation.

In this paper we consider a fuzzy optimization problem, i.e. a minimal set covering problem with expert data. The obtained bicriterion optimization problem is a specific compromised approach between expert and objective methods of optimization.

2 Preliminary Concepts

2.1 Classical set covering problem

Partitioning, covering and packing problems serve as a mathematical model for many theoretical and applied problems such as the coloring of graphs, construction of perfect codes and minimal disjunctive normal forms, drawing up of block-diagrams, information search, drawing up of traffic schedules, administrative division into zones and so on [2], [7], [8], [12].

Let us introduce some basic notions [2], [12]. Suppose that we are given the finite set \( R = \{r_1, \ldots, r_m\} \) and the family of its subsets \( S = \{S_1, \ldots, S_n\} \). Let \( S' = \{S_{j_1}, \ldots, S_{j_p}\}, 1 \leq p \leq n \), be some subfamily of the family \( S \). If each element \( r_i \) is contained in at most (at least) one of the sets \( S_j \) belong to \( S' \), then \( S' \) is called a packing (covering) of the set \( R \). A covering which is simultaneously a packing is called a partitioning of the set \( R \). Let \( A = ||a_{ij}||_{m \times n} \) be an incidence matrix of elements \( R \) and subsets \( S_j \): \( a_{ij} = 1 \) if \( r_i \in S_j \), and \( a_{ij} = 0 \) if \( r_i \notin S_j \). Each subfamily \( S' \) of the family \( S \) is given by means of the characteristic vector which has the component \( x_j = 1 \) if the subset \( S_j \) is contained in \( S' \), and \( x_j = 0 \) otherwise. If to each \( S_j \in S \) we assign a (positive) price \( c_j \), then partitioning, covering and packing problems take the form

1) \( \min_{A\pi=\bar{\pi}} (\bar{\pi}, \bar{x}) \); 2) \( \min_{A\pi \geq \bar{\pi}} (\bar{\pi}, \bar{x}) \); 3) \( \max_{A\pi \leq \bar{\pi}} (\bar{\pi}, \bar{x}) \);

where \( \bar{\pi} = (c_1, \ldots, c_n) \) is the price vector, \( \bar{x} = (x_1, \ldots, x_n) \) is the vector with components 0 and 1, and \( \bar{\pi} \) is the vector consisting of 1’s. Note that in many interesting problems \( c_j = 1, j = 1, \ldots, n \) (such is, for instance, the problem on finding a minimal dominating set in the graph), but this does not simplify the solution process of these problems.

2.2 On the most typical value (MTV) of a compatibility function with respect to a probabilistic or fuzzy measure

If data are represented in intervals, their distribution is obscure, they overlap and are described or obtained by an individual expert (insufficient expert data), then they are considered to be of combined nature. In that case, along with probabilistic-statistical uncertainty, there arises the so-called possibilistic uncertainty produced by an individual (expert) and demanding the application of fuzzy analysis methods. In such situations only probabilistic-possibilistic analysis can provide satisfactory results by using the fuzzy methods to be discussed below.

In describing such data functionally, in many real situations the property of additivity remains unrevealed for a measurable representation of a set and this creates an additional restriction. Hence, to study subjective insufficient expert data it is frequently better to use monotone estimators instead of additive ones.

We introduce the definition of a fuzzy measure (Sugeno, [35]) adapted to the case of a finite referential.

Definition 1 Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set and \( g \) a set function

\[ g : \mathcal{P}(X) \to [0, 1], \]

where \( \mathcal{P} \) is the power set of \( X \). We will say \( g \) is a fuzzy measure on \( X \) if it satisfies:

(i) \( g(\emptyset) = 0; g(X) = 1. \)
(ii) \( \forall A, B \subseteq X, \text{ if } A \subseteq B, \text{ then } g(A) \leq g(B). \)

A fuzzy measure is a normalized and monotone set function. It can be considered as an extension of the probability concept, where additivity is replaced by the weaker condition of monotonicity.

Let us, for example, consider three typical symptoms \( x_1, x_2, x_3 \), which indicate some illness \( y \). Let an expert (physician) provide objective-subjective data
using his/her wide experience and medical records of patients (another expert would certainly provide different data).

Assume that we have the following information: 80% of patients with illness $g$ exhibit the symptoms $x_1$ and $x_2$, 20% of them have the symptoms $x_1$ and $x_3$. This information can be written using the monotone instead of the additive, measure $g$ defined on the subsets of the set $X = \{x_1, x_2, x_3\}$ (Table 1).

Table 1: Distribution table showing the dual measures $g$ and $g^*$

<table>
<thead>
<tr>
<th>$A \subseteq X$</th>
<th>$g$</th>
<th>$g^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${x_1}$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>${x_2}$</td>
<td>0</td>
<td>0.8</td>
</tr>
<tr>
<td>${x_3}$</td>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>${x_1, x_2}$</td>
<td>0.8</td>
<td>1</td>
</tr>
<tr>
<td>${x_1, x_3}$</td>
<td>0.2</td>
<td>1</td>
</tr>
<tr>
<td>${x_2, x_3}$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>${x_1, x_2, x_3}$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

$g^*$ is called the dual measure of $g$ defined by $g^*(A) = 1 - g(\bar{A})$. Note that $g^*$ contains the same information as $g$ but is written in a different way.

Non-additive but monotone measures were first used in fuzzy analysis in the 80s by M. Sugeno [35].

The fuzzy integral is a functional which assigns some number or a compatibility value to each fuzzy subset when the fuzzy measure is already fixed. As known [35], the concept of a fuzzy integral makes it possible to condense information provided by a compatibility function and a fuzzy measure. Having the fuzzy measure determined, we can estimate a fuzzy subset by the most typical compatibility value ($MTV$). The $MTV$ is essentially different in content and significance from a probabilistic average even when a probabilistic measure is used instead of a fuzzy measure. The pre-image of the $MTV$ with respect to a compatibility function distinguishes from the universe the most typical representative values of the considered fuzzy subset.

As already known, fuzzy averages differ both in form and content from probabilistic-statistical averages and other numerical characteristics such as mode and median. Nevertheless, in some cases “nonfuzzy” (objective) and “fuzzy” (subjective) averages coincide [25], [35]. For a given set of fuzzy subsets with compatibility function values from the interval [0; 1], the fuzzy average determines the most typical representative compatibility value $ME – Monotone Expectation$.

The following fuzzy integral (based on the Choquet operator [6]) is the monotone expectation, which was defined by Bolaños et al. [5]:

### 2.2.1 Fuzzy measure and a monotone expectation

**Definition 2 ([5])** Let $g$ be a fuzzy measure on $X$ and $h : X \rightarrow \mathbb{R}_0^+$ a non-negative function. The monotone expectation of $h$ with respect to $g$ is

$$E_g(h) = \int_0^{+\infty} g(H_\alpha) \, d\alpha,$$

where $H_\alpha = \{x \in X \mid h(x) \geq \alpha\}$.

The monotone expectation always exists and it is finite for each $g$ and $h$. It is obvious that $E_g(\cdot)$ is a generalization of the mathematical expectation: that is what it becomes when the used fuzzy measure is a probability measure, that is,

$$E_P(h) = \int_X h \, dP,$$

where $P$ denotes a probability measure, $E_P$ – mathematical expectation.

Some of the most important properties of the monotone expectation are see in [24]-[27], [35], [40].

Since the monotone expectation is a generalization of the mathematical expectation, it can be questioned whether the former possesses some weaker property in relation to additivity than the latter. The following proposition gives an expression of the monotone expectation that permits us to analyse that question.

**Proposition 1 ([5])** If the values of a non-negative function $h$ are ordered as

$$h(x_1) \leq h(x_2) \leq \cdots \leq h(x_n),$$

then the monotone expectation of $h$ with respect to a fuzzy measure $g$ can be written as

$$E_g(h) = \sum_{i=1}^{n} h(x_i)(g(A_i) - g(A_{i+1})), $$

where $A_i = \{x_i, x_{i+1}, \ldots, x_n\}$, $i = 1, \ldots, n$, $g(A_{n+1}) = 0$. 

Thus, the monotone expectation is an additive functional for functions ordered equally.

We can also notice that \( E_g(h) \) is an average of the \( h \) function values weighted by

\[
p_i = g(A_i) - g(A_{i+1}), \quad i = 1, \ldots, n, \quad p_n = g(A_n).
\]

As

\[
\sum_{i=1}^{n} p_i = g(A_1) = g(X) = 1 \quad \text{and} \quad p_i \geq 0,
\]

the values \( p_i \) can be interpreted as the values of a probability function. Then \( E_g(h) \) is equivalent to the mathematical expectation of \( h \) with respect to that probability distribution.

The values \( p_i \) depend on the fuzzy measure \( g \) and the sets \( A_i \), which depend on \( h \) only in the order determined by its values. So we can say:

**Proposition 2** The monotone expectation of a non-negative function \( h \) with respect to a fuzzy measure \( g \) coincides with the mathematical expectation of \( h \) with respect to a probability that depends only on \( g \) and the ordering of the values of \( h \).

### 2.2.2 Fuzzy measure and the fuzzy expected value (FEV)

In this section, we discuss the main estimators of fuzzy statistics: the fuzzy expected value (FEV) of the population. The FEV determines MTV for a compatibility function.

**Definition 3** ([16]) Let \((X, F)\) be a measurable space, \( F \) be a Borel field (\( \sigma \)-algebra). A function \( g: F \rightarrow [0, 1] \) is called a fuzzy measure if the following conditions are fulfilled:

(i) \( g(\emptyset) = 0, g(X) = 1 \);

(ii) If \( A \subset B \), then \( g(A) \leq g(B) \);

(iii) If \( \{A_k\}/1 < k < \infty \) is a monotone sequence, \( \forall A_k \in F \), then \( \lim_{k \to \infty} g(A_k) = g \left( \lim_{k \to \infty} A_k \right) \).

\((X, F, g)\) is called a fuzzy measure space.

Let \( h \) be a compatibility function of some fuzzy subset of \( X \), \( h: X \rightarrow [0, 1] \) be an \( F \)-measurable function, i.e., \( \forall \alpha \in [0, 1] : H_\alpha = \{x \in X/h(x) \geq \alpha\} \in F \).

**Definition 4** ([16]) The FEV of the compatibility function \( h \) with respect to the fuzzy measure \( g \) is Sugeno’s integral over \( X \):

\[
\text{FEV}(h) = \int h \circ g(\cdot) \equiv \sup_{\alpha \in [0,1]} \{\alpha \wedge g(H_\alpha)\},
\]

where \( \wedge \) denotes a minimum of two arguments.

It clearly follows that the FEV somehow “averages” the values of the compatibility function \( h \) not in the sense of a statistical average but by cutting subsets of the \( \alpha \) level, whose values of a fuzzy measure \( g \) are either sufficiently “high” or sufficiently “low”.

Thus the FEV gives a concrete value of the compatibility function \( h \), this value being the most typical characteristic of all possible values with respect to the fuzzy measure \( g \), obtained by cutting off the “upper” and “lower” strips on the graph of \( g(H_\alpha) \).

Thus the information carried by \( h \) and \( g \) gets condensed in the FEV which is the most typical value of all compatibility values.

Consider the situation where \( X = \{x_1, x_2, \ldots, x_n\} \) is a finite set.

**Proposition 3** ([35]) If the values of a compatibility function \( h \) are ordered as

\[
h(x_1) \leq h(x_2) \leq \cdots \leq h(x_n),
\]

then the FEV of \( h \) with respect to a fuzzy measure \( g \) can be written as

\[
\text{FEV} = \max_i \{h(x_i) \wedge g(A_i)\} = \min_i \{h(x_i) \vee g(A_i)\},
\]

where \( A_i = \{x_i, x_{i+1}, \ldots, x_n\}, i = 1, \ldots, n \), and where \( \vee \) is a maximum of two arguments.

### 3 Aggregation by the monotone expectation in the set Covering Problem

Our further consideration concerns to a minimal fuzzy covering problem. Other problems of fuzzy partitioning and packing can be considered analogously. Let \( S = \{S_1, S_2, \ldots, S_n\} \) be some family of fuzzy subsets on \( R \). Denote the compatibility level \( \mu_{S_j}(r_i) \equiv \)
where a heuristic explanation of the positive ($p_{ij}$) and the negative ($n_{ij}$) discrimination measure is that $p_{ij}$ represents the accumulated belief that the element $S_j$ is more indicative (in the sense of covering) of an element $R_i$ than anyone of the remaining elements $R_l$ ($l = 1, \ldots, m, l \neq i$), while $n_{ij}$ represents the belief that an element $S_j$ is more indicative of not an element $R_i$, but of other elements $R_l$ ($l = 1, \ldots, m, l \neq i$) with respect to belief covering levels $b_{ij}$ of the fuzzy subsets $S_j$, $j = 1, \ldots, n$.

Suppose we are given some weights’ distribution

$$
\begin{array}{c}
\pi_j = \sum_{i=1}^{m} p_{ij} w_i, \\
v_j = \sum_{i=1}^{m} n_{ij} w_i,
\end{array}
(2)
$$

where $\pi_j$ and $v_j$ are called weighted average positive and the negative discrimination measures of the covering, respectively, for elements $S_j$, $j = 1, \ldots, n$.

Now, on the set $\{S_1, \ldots, S_n\}$ we construct a misbelief distribution of a covering, where both the positive and the negative discrimination measure ($\pi_j, v_j$) are taken into account;

$$
\delta_j = \nu m_{\text{small}}(\pi_j) + (1 - \nu)m_{\text{large}}(v_j),
(3)
$$

where $\nu$, $0 < \nu < 1$, is a weighted parameter which indicates preference of positive or negative discriminations.

The information content of $\delta_j$ is as follows: a covering level of misbelief in “the acceptance” of an element $S_j$.

Let $S' = \{S_j\}$, $k = 1, 2, \ldots, p; 1 \leq p \leq n$, be some fuzzy covering. It can be characterized by the
binary vector \( x_{S'} = (x_1, \ldots, x_n) \), where

\[
x_i = \begin{cases} 1, & \text{if the fuzzy subset } S_i \text{ is contained in } S', \\ 0, & \text{otherwise}. \\
\end{cases}
\]

Let us consider the misbelief distribution on \( x_{S'} \):

\[
\tilde{S}' \equiv (x_1, \ldots, x_n, \delta_1, \ldots, \delta_n).
\]

We say that the values \( x_j \) are chosen with some a priori information and we can consider some probability distribution on \( x_{S'} \):

\[
P_{S'} = \left( x_1, \ldots, x_n, p_1, \ldots, p_n \right).
\]

Thus for each fuzzy covering \( S' \) we have constructed the fuzzy misbelief distribution \((\delta_1, \ldots, \delta_n)\) on \( x_{S'} \) and the probability distribution \( P_{S'} \). Applying the method of fuzzy statistics which was presented in Subsection 2.2.1 [5], [25], [26], [28], [35], we define a fuzzy average value of \( S' \) as a monotone expectation [6] [25], (which here coincides with mathematical expectation)

\[
E_p(\tilde{S}') \stackrel{def}{=} \int_0^1 P_{S'}(\mu_{S'} \geq \alpha)d\alpha = \sum_{j=1}^n x_j \delta_j p_j.
\]

Note that the value \( E_p(\tilde{S}') \) is an average measure of misbelief in a fuzzy covering. Minimizing the average misbelief in the fuzzy covering \( \tilde{S}' \), we obtain the criterion

\[
\sum_{j=1}^n x_j \delta_j' \rightarrow \min,
\]

where \( \delta' = (\delta_1 p_1, \ldots, \delta_n p_n) \).

Finally, the minimal fuzzy covering problem is reduced to a bicriterion problem of the type (minsum-minsum) [12] for an ordinary covering with the target functions

\[
f_1 = \sum_{j=1}^n c_j' x_j \rightarrow \min
\]

(minimization of an average price)

\[
(\delta_j' = c_j p_j),
\]

\[
f_2 = \sum_{j=1}^n \delta_j' x_j \rightarrow \min
\]

(minimization of an average misbelief).

If \( X \) is the set of all Boolean vectors satisfying the conditions of the fuzzy covering problem, then by considering the scalar optimization problem

\[
\lambda f_1 + (1 - \lambda)f_2 \rightarrow \min,
\]

\[
(x_1, \ldots, x_n) \in X, \quad \lambda \in (0, 1),
\]

where

\[
X = \{x_{S'} \in \{0, 1\}^n \mid \tilde{S}' \subset \tilde{S}, \tilde{S}' \text{ is the covering} \} \equiv \{\bar{x} \in \{0, 1\}^n \mid A\bar{x} \geq \bar{e}\}
\]

and \( \lambda \) is a weighted parameter, we can find, in the general case, some Pareto optima [12].

An aggregation by the FEV in Minimal Fuzzy Covering Problem is the problems of the future investigation.

4 Conclusion

We apply the methods of fuzzy statistics [23], [25], [28] to the considered discrete optimization problems with fuzzy data. In an appropriate manner we introduce the definitions of positive and negative discriminations of expert knowledge of optimization problem parameters, i.e. parameters of possible solutions and alternatives (candidates). We thereby determine a fuzzy distribution of misbelief on the set of alternatives.

As a result we obtain a bicriterial discrete optimization problem which is solved by the method of linear convolution of criteria. The scalar problem is solved by the search tree algorithm from [7].

The obtained bicriterial optimization problem is a specific compromised approach between the expert
(fuzzy) and the objective (probabilistic) method of optimization of decision-making, where both the minimization of average misbelief in alternatives and the minimization of an average price for alternatives are taken into account. The constructed approach (minimal fuzzy covering) to the solution of the discrete optimization problem with data of combined (expert-objective) nature can be regarded as more trustworthy from the standpoint of application than the classical optimization methods.

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