Predicting the dynamic response of slab track with continuous slabs under moving load

TRAIAN MAZILU
Department of Railway Vehicles,
University Politehnica of Bucharest,
Splaiul Independentei, 313, Bucharest 060032, Romania
trmazilu@yahoo.com

Abstract: - The paper discusses the problems of modeling and simulation of the dynamic response of slab track with continuous slabs under moving loads. The equations of motion are solved following three steps. First, the equations are transformed from the time-domain to the frequency-domain via the Fourier transformation method. Second, the solution of the transformed equations is obtained using the properties of the Green’s function of the differential operator associated to the aforementioned equations. Finally, the solution in the time-space domain results from the inverse Fourier transform. The response of the slab track to the stationary harmonic load is analyzed. The displacement pattern of the rail for various frequencies and velocities of the load is presented.

Key-Words: - slab track, rail, bending wave, Doppler effect

1 Introduction

This paper is dedicated to the analysis of the dynamic response of slab track with continuous slabs due to the moving harmonic load, which moves uniformly along the track. As it is known, the slab track is applied for high-speed lines [1] and for urban railway environment [2] - and this kind of problem represents the starting point for the studies related by the wheel/rail interaction.

Usually, the structure of the slab track is composed of a massive concrete slab, into which the rails are embedded by means of Corkelast. Assuming that the two rails are symmetrically loaded, only half-track is required for modelling. Consequently, the slab track model is reduced to an infinite homogenous structure consisting of two beams continuously supported by elastic layers. The upper beam describes a rail, the lower one models the slab, while the two elastic layers reflect the properties of the rail pad and the track subsoil. Similar models were used to study the slab track response to a moving load [3, 4] or the interaction between a moving vehicle and the slab track [5, 6].

In order to solve the issue of the slab track response to moving load, many methods have been proposed. The direct method treats the input force as a boundary condition for the problem. More precisely, the input force is described for the right semi-infinite structure as a boundary condition on its left end. For the left semi-infinite structure, the input force is described as a boundary condition on its right end. For the case of non-moving load, the entire structure has the property to be symmetric. Due to symmetry, it is enough to analyse only the right semi-infinite structure.

The Fourier transformation method treats the load as a part of the differential equation. The governing differential equations of the slab track are transformed to the wave number-frequency domain. Then, the transformed equations are simplified and transformed back to the space-time domain using the results of contour integration from the theory of complex variables or the inverse discrete Fourier transform (numerical method).

The method of coupling in the wave number-frequency domain splits the model of the slab track into two structures, the rail and the slab on elastic foundation. These two structures are coupled via rail pad, which is uniformly distributed longitudinally. Applying this method, the frequency response functions of the two structures has to be separately found in the wave number-frequency domain.

In this paper, a different method is proposed. Starting from the governing differential equations, the Fourier transform is performed to obtain the transformed equations in the frequency-domain. Then, the transformed equations are solved using the Green’s function of the differential operator belonging to the transformed equations. Finally, the inverse Fourier transform is utilized to find the solution of the problem. This method is analytical one and seems to be simpler.
2 Problem Formulation

The mechanical model of the slab track is presented in fig. 1. The model consists of double Euler-Bernoulli beams coupled by a Winkler layer, which are supported by a Winkler foundation as well. It has to emphasize that the model of the Euler-Bernoulli beam may be applied as long as the cross-sectional dimensions are small compared to the bending wavelength \[7\].

The parameters for the slab track model are as following: the mass per length unit \(m_{1,2}\) (the index 1 for the upper beam and the index 2 for the under beam) and the bending stiffness \(EI_{1,2}\). The two Winkler foundations have the elastic constants \(k_{1,2}\) per length unit and the viscous damping factors \(c_{1,2}\) per length unit.

Assuming that the slab track is subject to a moving harmonic load \(Q(t) = Q_0 \cos \Omega t\) with the amplitude \(Q_0\) and the angular frequency \(\Omega\) (time), the differential equations of motion may be written in matrix form as

\[ L_{x,t} \{w(x,t)\} = \{q(x,t)\}, \quad (1) \]

where \(L_{x,t}\) stands for the matrix differential

\[
L_{x,t} = \begin{bmatrix}
D_{B1} + c_1 \frac{\partial}{\partial t} + k_1 & -c_1 \frac{\partial}{\partial t} - k_1 \\
-c_1 \frac{\partial}{\partial t} - k_1 & D_{B2} + (c_1 + c_2) \frac{\partial}{\partial t} + k_1 + k_2
\end{bmatrix},
\]

which includes the differential operators of the Euler-Bernoulli beams

\[
D_{B1,2} = EI_{1,2} \frac{\partial^4}{\partial x^4} + m_{1,2} \frac{\partial^2}{\partial t^2},
\]

\(\{w(x,t)\} = [w_1(x,t) \ w_2(x,t)]^T\) is the column vector of the rail and slab displacements with \(x\) the coordinate along the track, and \(\{q(x,t)\} = [Q_0 \cos \Omega t \delta(x-Vt) \ 0]^T\) is the column vector of the forces on the track with \(\delta(.)\) is the Dirac delta functions and \(V\) speed.

The boundary conditions are

\[
\lim_{|x| \to \infty} \{w(x,t)\} = [0 \ 0]^T. \quad (2)
\]

By introducing the corresponding complex variable so that

\[
\{w(x,t)\} = \text{Re}\{\tilde{w}(x,t)\}, \quad (3)
\]

and setting \(Q_0 = 1\) N, the equation (1) becomes

\[
L_{x,t} \{\tilde{w}(x',t)\} = \left[e^{i\Omega t} \delta(x-Vt) \ 0\right]^T, \quad (4)
\]

where \(i^2 = -1\).

At this stage, it can replace the fixed frame of axis, i.e. \((x, t)\), by the moving frame of axis, i.e. \((x' = x-Vt, t)\). The transformed equation reads

\[
L_{x',t} \{\tilde{w}(x',t)\} = \left[e^{i\Omega t} \delta(x' - Vt) 0\right]^T, \quad (5)
\]

where the matrix operator \(L_{x',t}\) is obtained from \(L_{x,t}\) taking

\[
\frac{\partial^n}{\partial x^n} = \frac{\partial^n}{\partial x'^n}, \quad \frac{\partial^n}{\partial t^n} = \left(\frac{\partial}{\partial t} - V \frac{\partial}{\partial x'}\right)^n. \quad (6)
\]

3 Problem Solution

Next, the attention is paid on the steady state behavior of the slab track under moving harmonic load.

In order to find the solution, the Fourier transform is applied with respect to time

\[
L_{x',\omega} \{\tilde{w}(x',\omega)\} = 2\pi \left[\delta(x') \delta(\omega - \Omega) 0\right]^T, \quad (7)
\]

where

\[
\{\tilde{w}(x',\omega)\} = \int_{-\infty}^{\infty} \{\tilde{w}(x',t)\} e^{-i\omega t} dt,
\]

\(L_{x',\omega} = F[L_{x',t}]\),
$F[.]$ stands for the Fourier transform and $\omega$ is the coordinate of the frequency-domain.

The adjoint matrix operator $L^*_{x',\omega}$ may multiply the equation (7)

$$\text{diag}(D,D)[\mathbf{W}(x',\omega)] = 2\pi L^*_{x',\omega}\left[\delta(x')\delta(\omega-\Omega)\right] 0^T,$$

(8)

where the differential operator

$$D = \sum_{n=0}^{8} a_n(\omega, V) \frac{d^n}{dx^n}$$

has the complex coefficients depending on the track’s parameters, the angular frequency $\omega$ and the speed $V$.

The solution of the equation (8) may be expressed using the Green’s function of the $D$ differential operator

$$\{\mathbf{W}(x',\omega)\} = 2\pi \delta(\omega-\Omega) \int_0^\infty G(x',\xi,\omega)L^*_{x',\omega}\left[\delta(\xi)\right] 0 d\xi$$

(9)

where $G(x',\xi,\omega)$ stands for the Green’s functions of the $D$ differential operator which verifies the equation

$$DG(x',\xi,\omega) = \delta(x' - \xi)$$

(10)

and the boundary condition

$$\lim_{|x'| \rightarrow \infty} G(x',\xi,\omega) = 0$$

(11)

due to the damping of the track.

The Green’s function of the $D$ operator is obtained using its outstanding features [8]. In fact, the operator $D$ has the following solutions

$$y_i(x') = A_i e^{\lambda_i x'}$$

(12)

where $\lambda_i = \lambda_i(\omega, V)$ satisfies the characteristic equation of the $D$ operator. Actually, $\lambda_i$ describes the bending wave, which propagates from the moving load through the track structure. This wave propagates from the moving load to the right when $\text{Im} \lambda_i > 0$, and from the moving load to the left when $\text{Im} \lambda_i < 0$. The bending wave attenuates when $\lambda_i$ is a complex quantity. According to the boundary conditions, the attenuated wave decreases with distance ($\text{Re} \lambda_i > 0$ for $x' \rightarrow -\infty$ and $\text{Re} \lambda_i < 0$ for $x' \rightarrow \infty$). Finally, the bending wave is an evanescent one when $\lambda_i$ is a real quantity. In addition, the evanescent wave has to propagate following the same rule as the attenuated one, according to the boundary conditions.

In the light of the preceding considerations, the Green’s function of the $D$ operator has the form

$$G^-(x',\xi,\omega) = \sum_{i=1}^{s} A_i(\xi) e^{\lambda_i x'} \text{ for } -\infty < x' < \xi, \quad (13)$$

$$G^+(x',\xi,\omega) = \sum_{i=s+1}^{8} A_i(\xi) e^{\lambda_i x'} \text{ for } \xi < x' < \infty, \quad (14)$$

with $\text{Re}(\lambda_i) > 0$ for $i = 1 \div s$ and $\text{Re}\lambda_i < 0$ for $i = s+1 \div 8$. On the other hand, the Green’s function is continuous in $x' = \xi$ and its first 6 derivates are continuous as well

$$G^-((\xi,\xi,\omega),\xi,\omega) = G^+(\xi,\xi,\omega)$$

(15)

Further on, the $7^{th}$ derivate of the Green function has a discontinuity in $x' = \xi$

$$\frac{d^7 G^-((\xi + 0,\xi,\omega))}{dx'^7} = \frac{d^7 G^+(\xi - 0,\xi,\omega)}{dx'^7} = \frac{1}{a_8}$$

(17)

All these conditions lead to the next matrix equation

$$\begin{bmatrix}
1 & 1 & \ldots & 1 \\
\lambda_1 & \lambda_2 & \ldots & \lambda_8 \\
\lambda_1^2 & \lambda_2^2 & \ldots & \lambda_8^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^7 & \lambda_2^7 & \ldots & \lambda_8^7
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
\vdots \\
X_8
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
-a_8^{-1}
\end{bmatrix}$$

(18)

where $X_i = A_i(\xi)\exp(\lambda_i \xi)$ for $i = 1 \div s$ and $X_i = -A_i(\xi)\exp(\lambda_i \xi)$ for $i = s+1 \div 8$. Obviously, the matrix from Eq. (18) has the Vandermon determinant and in fact, all Cramer’s determinants are Vandermon determinants, as well. Eq. (18) has the following solution

$$X_i = \frac{1}{a_8 \prod_{k \neq i} (\lambda_k - \lambda_i)}$$

(19)

and finally, the Green’s function of the $D$ operator is obtained
Next, the dynamic response of a particular slab track under a moving load is presented. Physical parameters of the track model used in these computations are as follows: \( m_1 = 60 \text{ kg/m}, \ EI_1 = 6.42 \text{ MNm}^2, \ m_2 = 1750 \text{ kg/m}, \ EI_2 = 274 \text{ MNm}^2, \ k_1 = 52 \text{ MN/m}^2, \ c_1 = 7 \text{ kNs/m}^2, \ k_2 = 60 \text{ MN/m}^2 \) and \( c_2 = 40 \text{ kNs/m}^2 \).

Inserting the Green’s function into the equation (9) and performing the inverse Fourier transform, the solution reads

\[
\begin{align*}
\bar{w}_1(x', t) &= e^{i \xi t} \sum_{n=1}^{8} b_{1n} \frac{1}{a_8} \prod_{k=1 \text{ or } 2} \left( \lambda - \lambda_i \right) \Theta \left( x' - x \right) \Theta \left( \xi - x \right) \\
\bar{w}_2(x', t) &= e^{-i \xi t} \sum_{n=1}^{8} b_{2n} \frac{1}{a_8} \prod_{k=1 \text{ or } 2} \left( \lambda - \lambda_i \right) \Theta \left( x' - x \right) \Theta \left( \xi - x \right)
\end{align*}
\]

for \( x' > 0 \)

where \( \lambda_i = \lambda_i (\Omega, V) \) and the coefficients

\[
\begin{align*}
b_{10} &= k_1 + k_2 - m_2 \Omega^2 + i \Omega (c_1 + c_2), \\
b_{11} &= V (c_1 + c_2 + 2i \Omega m_2), \\
b_{12} &= m_2 V^2, \\
b_{13} &= 0, \ b_{14} = EI_2, \\
b_{20} &= k_1 + i \Omega c_1, \\
b_{21} &= c_1 V.
\end{align*}
\]

It may observe that the column vectors from the right hand represent the receptances of the track in moving frame.

Figure 2 presents the receptances of the track, i.e. the rail and the slab receptances, which have been computed at the point of a stationary harmonic load \((V = 0, x = x' = 0)\). The response of the track has two peaks at 29 and 150 Hz because the rail and the slab vibrate as a discrete system with two degrees of freedom. At the first resonance frequency, the rail and the slab are in phase and then, they vibrate in anti-phase. The first peak belongs to the slab resonance, and the second one is the effect of the resonance of the rail. In fact, the two frequencies of resonance may be approximately calculated using formula from the single-degree-of-freedom system consisting of an equivalent mass and a spring with...
equivalent stiffness due to the high difference between them

\[ f_{1,2} = \frac{1}{2\pi} \sqrt{\frac{k_{1,2}}{m_{1,2}}} . \]

The receptance of the rail is significantly higher than the receptance of the slab due to its low inertia and the elasticity of the rail pad. The diagram is similar with the results from the preceding related researches [3, 4, 5].

Figure 3 shows the influence of the speed of the moving harmonic load on the receptance of the rail \((x' = 0)\). It may be seen that by increasing the speed of the moving harmonic load, the resonance frequencies of the track decrease. In addition, the receptance of the rail lows around resonance frequencies due to the speed of the harmonic excitation.

The displacement pattern of the rail is presented in figs. 4 and 5 for the time moment \(t = 0\) when the harmonic load moves along the rail at 100 m/s. Figure 4 corresponds to the load frequency of 20 Hz. The displacement pattern is localized near load, exponentially decaying with the distance from the loading point because the structure does not generate propagating waves at this frequency.

Figure 5 displays the rail displacement for the load frequency of 200 Hz. The waves propagating in front and behind of load may be clearly observed.

Figure 6 shows the displacement of the fixed point belonging to the section of the rail situated at 3m from the frame, when a unitary harmonic load with the frequency of 200 Hz travels along the rail at 100 m/s. The displacement of the loading point is presented for comparison. The two displacements have the same value only when the moving load passes over the section of the fixed point. The history of the fixed point is later than the loading point’s except in the joining moment because the rail has propagating wave. For instance, it can observe that all peaks of the fixed point are later than the corresponding ones of the loading point. Consequently, the fixed point has a frequency higher than the loading point’s one before the joining and then, after joining, this trend reverses. In other words, this is the Doppler effect. In fact, the consequence of the Doppler effect may be seen even in figure 5 where the wave propagating in front of the load is shorter than that propagating behind the load.

Finally, the displacement of the rail at moving point is plotted in figure 7 as a function of the load velocity for the case of the non-oscillating moving load. The maximum of the rail displacement corresponds to the critical load velocity, around 380 m/s. The figure also shows that the velocity effect is negligible up to 100 m/s because the static and moving load models give approximately the same solution.
4 Conclusion
The steady-state response of a model for a slab track with continuous slabs to moving harmonic load has been studied. To this end, the equations of motion from the model of the slab track have been solved using the method based on the Fourier transform with respect to time and the Green’s function of the resulted differential operator.

A study of the slab track response to stationary harmonic load has been accomplished. In addition, the analysis of the displacement pattern of the rail for various frequencies and velocities of the load has been presented.

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References: