Fixed Point Technique for a class of Nonlinear Systems and Application to Stochastic Resonance

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Abstract: In this paper, we study a nonlinear system under regime switching and subject to an environmental noise. We will considered some more general conditions for the coefficient functions and prove a result on the existence using the Schauder’s fixed point theorem extended some similarly results on linear systems. Also, we study the application of these system to control the electronic circuits using the benefit of stochastic resonance

Key-Words: stochastic differential equation with Markovian switching, pathwise uniqueness, stopping times, stochastic resonance

1 Introduction

The hybrid systems driven by continuous-time Markov chains have been used to model many practical systems where they may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. The hybrid systems combine a part of the state that takes values continuously and another part of the state that takes discrete values. The term describing the influence of interest rates was modeled by a finite-state Markov chain to provide a quantitative measure of the effect of interest rate uncertainty on optimal policy (see Bensoussan 2000, Bouks 1993, Ghosh 1993, Hu 2000, etc.). One of the important classes of the hybrid systems is the stochastic differential equations with Markovian switching (SDEwMSs) (see Ji, 1990, Mao, 1999, Mao, 2006).

In this paper, we shall discuss the existence and uniqueness of the solution on a general nonlinear stochastic differential equations with Markovian switching of the form

\[ dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dw(t) \]

(1)

This equation can be regarded as the result of the following N equations:

\[ dx(t) = f(x(t), t, i)dt + g(x(t), t, i)dw(t), \quad 1 \leq i \leq N \]

(2)

switching from one to the others according to the movement of the Markov chain. We consider \( r(t), t_0 \leq t \leq T \), be a right-continuous Markov chain on the probability space taking values in a finite state space \( S = \{1, 2, \ldots, N\} \) with generator \( \Gamma = (\gamma_{ij})_{i,j=1,N} \) given by

\[
P[r(t + \Delta) = j | r(t) = i] = \\
= \begin{cases} 
\gamma_{ij} + o(\Delta), & \text{if } i \neq j \\
1 + \gamma_{ij} + o(\Delta), & \text{if } i = j
\end{cases}
\]

where \( \Delta > 0 \) here \( \gamma_{ij} \) is transition rate from \( i \) to \( j \) if \( i \neq j \) while \( \gamma_{ii} = -\sum_{j \neq i} \gamma_{ij} \). We assume that the Markov chain \( r(t) \) is independent of the Brownian motion \( w(t) \). It is well known that almost every sample path of \( r(t) \) is a right-continuous step function and \( r(t) \) is ergodic. It is known that almost every sample path of \( r(t) \) is a right-continuous step function with a finite number of simple jumps in any finite subinterval of \( R_+ \).

2 Main results

Let \( \{w(t)\}, 0 \leq t_0 \leq t \leq T \) \((T \in R_+)\), denote a Wiener process defined on the probability space \((\Omega, F, P)\). Suppose \( \{F_t: t_0 \leq t \leq T\} \) is a non-anticipating family of sub-\( \sigma \)-algebras of \( F \) with respect to the Wiener process \( w_t \).
First, in a some similar way as (Athanassov, 1990, Constantin A., 1996, Negrea 2003), we give the following lemma:

**Lemma 2.1.** Let \( u(t) \) a continuous, positive function on \( a < t \leq b \) (\( a < b \), two real numbers), having nonnegative derivative \( u'(t) \in L(a,b) \). Let \( v(t) \) a continuous, nonnegative functions for \( a < t \leq b \) such that \( v(t) = o(u(t)) \) as \( t \to a^+ \) and \( v(t) \leq f_a + \frac{u'(s)}{u(t)} v(s)ds \), \( \forall a < t \leq b \).

Then \( v(t) \equiv 0 \) on \( a \leq t \leq b \).

The basic idea in proving the existence of the solution in a SDEwMS is to analyse a standard SDE on each interval \([t_k, t_{k+1}]\), \( k \geq 0 \), where \( \{t_k\}_{k \geq 1} \) are the associated stopping times of the Markov process \( \{r(t)\}_{t \in [0, T]} \). Some results on the existence and uniqueness of solutions and on the convergence of successive approximations for stochastic differential equations, assuming the existence of a function \( u \) controlling the growth and the continuity of \( f \) and \( g \) (as in Constantin A., 1996, Negrea 2003, Constantin I., 2004, etc.) generalizing to the setting of stochastic differential equations driven by Brownian motion a result of Athanassov (see Athanassov 1990).

Now, we recall the SDEwMS (1) and we consider the equivalent integral equation

\[
x(t) = x(t_0) + \int_{t_0}^{T} f(x(s), s, r(s))ds + \int_{t_0}^{T} g(x(s), s, r(s))dW_s
\]

with \( f, g : \Omega \times R \times [t_0, T] \times R \) and the following hypotheses:

(H1). \( f, g \) are \( B \otimes P \otimes B \) measurable functions;
(H2). \( f(0, \cdot, \cdot) \in L^2([t_0, T], R) \) and \( g(0, \cdot, \cdot) \in M^2([t_0, T], R) \);
(H3). there exists \( u(t) \) a continuous, positive and derivable function on \( t_0 < t \leq T \) with \( u(t_0) = 0 \), having nonnegative derivate \( u'(t) \in L(t_0, T) \) such that

\[
|f(x, t, i) - f(y, t, i)|^2 \land |g(x, t, i) - g(y, t, i)|^2 \leq \frac{u'(t)}{3u(t)}|x - y|^2,
\]

for all \( x, y \in R, \ t_0 < t \leq T, \ i \in S; \)

(H4). with the same function \( u(t) \) as above we have

\[
|f(x, t, i)|^2 \land |g(x, t, i)|^2 \leq u'(t)(1 + |x|^2);
\]

(H5). \( x(t_0) = x_0 \) is a given \( F_{t_0} \)-measurable random variable such that \( E[|x_0|^2] < \infty \).

**Theorem 2.2.** Let be \( f \) and \( g \) satisfying the above hypotheses and \( x_0 \in L^2(\Omega, F_{t_0}, P, R) \), then there exists a unique solution \( x \in L^2([t_0, T], R) \) which satisfies the equation (1) for \( t_0 \leq t \leq T \).

**Proof.** Uniqueness. Let be \( x(t) \) and \( y(t) \) two solutions in \( L^2([t_0, T], R) \) of the equations (5). We have

\[
E[x(t) - y(t)]^2 \leq 3\{E[\int_{t_0}^{t} |f(x(s), s, r(s)) - f(y(s), s, r(s))|^2ds + \int_{t_0}^{t} |g(x(s), s, r(s)) - g(y(s), s, r(s))|^2ds]\} \leq \int_{t_0}^{t} u(s)E[|x(s) - y(s)|^2]ds
\]

and from the Lemma 2.1. with \( v(t) = E[|x(t) - y(t)|^2] \) yields that \( x(t) \equiv y(t) \) on \( [t_0, T] \).

Existence. Recall that almost every simple path of \( r(\cdot) \) is a right continuous step function on \([t_0, T]\). Therefore, is a sequence \( \{t_k\}_{k \geq 1} \) of stopping times such that for almost every \( \omega \in \Omega \) there is a finite \( K = K(\omega) \) for \( t_0 < t_1 < \ldots < t_k = T \) and \( t_k = T \) if \( k > K \) and \( r(t) = r(t_k) \) for \( t_k \leq t < t_{k+1} \) for all \( k \geq 1 \).

We first consider the equation (1) on \( t \in (t_0, t_1) \), which becomes

\[
dx(t) = f(x(t), t, r_0)dt + g(x(t), t, r_0)dW_t
\]

with the initial data \( x_0 \) and \( r_0 \).

We consider the operator

\[
T : C_a([t_0, t_1]) \rightarrow C_a([t_0, t_1])
\]

by

\[
T x(t) = x(t_0) + \int_{t_0}^{t_1} f(s, x(s), r_0)ds + \int_{t_0}^{t_1} g(s, x(s), r_0)dW(s)
\]

As in [4] we define the set

\[
B = \{x \in C_a([t_0, t_1]) : \|x(t)\|^2 \leq m(t), t_0 \leq t \leq t_1\}
\]

where \( m(t) \) is the maximal solution of the differential equation

\[
m'(t) = 6Ku(t)m(t), \ t \in [t_0, t_1]
\]

with the initial condition

\[
m(0) = Q = 3 \sup_{t \in [t_0, t_1]} |x_0(t)|^2 + 3KMt_1
\]
with
\[ M = \max \{ \| f(t,0,r_0) \|^2, \| g(t,0,r_0) \|^2 \}. \]

We deduce that
\[
\begin{align*}
\|Tx(t)\| & \leq \|x_0(t)\|^2 + x_0(t) + \int_{t_0}^{t_1} \| f(s,x(s),r_0) \|^2 ds \\
& \quad + \int_{t_0}^{t_1} \| g(s,x(s),r_0) \|^2 dw(s) \\
& \leq \sup_{t \in [0,t_1]} \| x_0(t) \| + \sqrt{2} \int_{t_0}^{t_1} u'(s)(\| x(s) \|^2) ds + M t_1 \frac{1}{2} \\
& \quad + \sqrt{2} K \int_{t_0}^{t_1} u'(s)(\| x(s) \|^2) ds + M t_1 \frac{1}{2} \\
& \leq \sup_{t \in [0,t_1]} \| x_0(t) \| + \sqrt{2} K \left\{ \int_{t_0}^{t_1} u'(s)(\| x(s) \|^2) ds + M t_1 \right\} \frac{1}{2} \\
& \quad + \sqrt{2} K \left\{ \int_{t_0}^{t_1} u'(s)(\| x(s) \|^2) ds + M t_1 \right\} \frac{1}{2} \\
& \quad + 3 \sup_{t \in [t_0,t_1]} \| x(t) \|^2 \\
& \quad + 6 K^2 \int_{t_0}^{t_1} u'(s)(m(s)) ds + 6 K^2 M t_1 \frac{1}{2} \\
& \quad + 6 K^2 \int_{t_0}^{t_1} u'(s)(m(s)) ds + 6 K^2 M t_1 \frac{1}{2} \\
& \quad + Q + 6 K^2 \int_{t_0}^{t_1} u'(s)(m(s)) ds + 6 K^2 M t_1 \frac{1}{2} \\
& \quad + 6 K^2 \int_{t_0}^{t_1} u'(s)(m(s)) ds = m(t), \quad t_0 \leq t \leq t_1
\end{align*}
\]

and for \( x \in B \) we obtain that
\[
\|Tx(t)\|^2 \leq \sup_{t \in [0,t_1]} \| x(t) \| + \sqrt{2} K \left\{ \int_{t_0}^{t_1} u'(s)(\| x(s) \|^2) ds + M t_1 \right\} \frac{1}{2} \\
\begin{align*}
& \quad + \sqrt{2} K \left\{ \int_{t_0}^{t_1} u'(s)(\| x(s) \|^2) ds + M t_1 \right\} \frac{1}{2} \\
& \quad \leq 3 \sup_{t \in [t_0,t_1]} \| x(t) \|^2 \\
& \quad + 6 K^2 \int_{t_0}^{t_1} u'(s)(m(s)) ds + \\
& \quad + 6 K^2 \int_{t_0}^{t_1} u'(s)(m(s)) ds + 6 K^2 M t_1 \\
& \quad = Q + 6 K^2 \int_{t_0}^{t_1} u'(s)(m(s)) ds + \\
& \quad + 6 K^2 \int_{t_0}^{t_1} u'(s)(m(s)) ds = m(t), \quad t_0 \leq t \leq t_1
\end{align*}
\]

We proved so that \( T(B) \subseteq B \), and it is easy to see that the set \( B \) is a closed, bounded and convex subset of the Banach space \( C_a([t_0,t_1], ||.|||) \), with
\[
||| (x,y) ||| = \sqrt{\| x \|^2 + \| y \|^2}
\]

where \( \| x \|^2 = E[\sup_{0 \leq t \leq 1} | x(t) |^2] \).

On the other hand we have that
\[
\| Tx(t) - Tx(s) \|^2 \leq 6 K^2 \int_s^t u(s)(m(s)) ds + 6 K^2 \int_s^t u(s)(m(s)) ds + \\
+ 3 \sup_{t \in [0,t_1]} \| x(t) - x(s) \|^2 + 6 K^2 M(t - s),
\]

\( t_0 \leq s \leq t \leq t_1 \) and thus the set \( T(B) \) is equicontinuous.

In a similar way we prove that for \( x, y \in B \) we have
\[
\| Tx(t) - Ty(t) \| \leq \left\{ \int_{t_0}^{t_1} \| f(s,x(s),r_0) - f(s,y(s),r_0) \|^2 ds \right\} \frac{1}{2} + \\
+ \left\{ \int_{t_0}^{t_1} \| g(s,x(s),r_0) - g(s,y(s),r_0) \|^2 ds \right\} \frac{1}{2},
\]

\( t_0 \leq t \leq t_1 \). From \( H_3 \) and the continuity of \( f(s,x,s) \) and \( g(s,x,s) \) in \( x \) on \( L^2 \) we deduce by Lebesgue convergence theorem that \( T \) is continuous.

An application of Schauder’s fixed point theorem enables us to deduce that \( T \) has a fixed point in \( B \), thus equation (1) has a solution on \( [t_0,t_1] \).

We repeat this procedure and we can see that the equation has a solution \( x(t) \) on \( [t_0,T] \).

Remark. It is easy to see that our results extend more classical results (with lipschitz conditions for example, (see Mao 1999, Mao 2006, etc.). The problems of discontinuities in the stopping times \( t_i \), \( i = 1, 2, \ldots, N \) appears in more applications when are some changes in the behavior of physical phenomena modeling from the adapted process \( x(t) \).

3 Comments and Applications

Noise in dynamical system is usually considered a nuisance. However, in certain nonlinear systems, including electronic circuits and biological sensory systems, the presence of noise can enhance the detection of weak signals. The phenomenon is termed stochastic resonance and is of great interest for electronic instrumentation (see Gammaitoni, 1998, McNamara, 1989, Negrea, 2007a, Negrea, 2007b).

The essential ingredient for the stochastic resonance is a nonlinear dynamical system, which typically has a period signal and noise at the input and output that is a function of the input as well as the internal dynamics of the system. The nonlinear component of the dynamical system is sometimes provided.
by a threshold which must be crossed for the output to be changed or detected. A nonlinear system is essential for stochastic resonance to exist, since in a system that is well characterized by linear response theory, the signal-to-noise ratio at the output must be proportional to the signal-to-noise ratio at the input.

Engineers have normally sought to minimize the effect of noise in electronic circuits and communication systems. Today, however, it is acknowledged that noise or random motion is beneficial in breaking up the quantization pattern in a video signal, in the dithering of analog to digital converters, in the area of Brownian ratchets, etc.

A model of one-dimensional nonlinear system that exhibits stochastic resonance is the damped harmonic oscillators with the Langevin equation of motion (see Gammaitoni 1998):

\[ m\ddot{x}(t) + \gamma \dot{x}(t) = -\frac{dU(x)}{dx} + \sqrt{D}\xi(t) \]

This equation describes the motion of a particle of mass \( m \) moving in the presence of friction \( \gamma \). The restoring force is expressed as the gradient of some bistable or multi-stable potential function \( U(x) \). In addition, there is an additive stochastic force \( \xi(t) \) with intensity \( D \), and, in generally, it supposed as been a white Gaussian noise.

In the case of symmetrical bistable system, the potential function is a simple symmetric function and, adding a period signal and considering case of time dependent system (see Berglund 2002, Gammaitoni, 1998)

\[ U(x, t) = U(x) - Ax \sin(\omega_s t) = -\frac{a}{2}x^2 + \frac{b}{4}x^4 - Ax \sin(\omega_s t) \]

where \( A \) and \( \omega \) are the amplitude and the frequency of the periodic signal, respectively.

In the last years, engineers used the asymmetrical bistable system (see Herman 2002, Imkeller 2001), when the potential function has the expression:

\[ U(x, t) = \begin{cases} U(x) - Ax \sin(\omega_s t), & \text{if } t \in [0, \frac{1}{2}] \\ U(-x) - Ax \sin(\omega_s t), & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases} \]

\[ U(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4 \]

or more general model with \( k \) switching times (see Imkeller, 2001)

\[ U(x, t) = \sum_{k \geq 0} U(x)1_{[k,k+0.5)} + U(-x)1_{[k+0.5,k+1)} \]

About these models there is a simple observation: we can not say exactly if the external perturbation is present just at the discrete moments. A continuous model appears as more adequate. This approach is possible just using the theory of stochastic differential equations.

A problem, which frequency appear in practice, is the value of initial state \( x_0 \). This value is "proposed" but in some non-standard external conditions, this make an discontinuity of the simple path of the process \( \{ x(t) \} \) and this phenomena is repeating at any stopping time \( t_i, (i = 1, 2, \ldots, N) \) and we will have new discontinuities at these time moments. In applications, we have a right-continuity for the Markov process but just a left-continuity for the process \( \{ x(t) \} \) (we have a very short delay at the stopping times). Therefore, is necessary to consider some general coefficient functions and good stability properties of the solutions. On the other hand, the stochastic resonance make possible a control of the electronic circuits in some external stochastic perturbations by controlling the adapted process \( \{ x(t) \} \).

References:


