Non-conventional Interpretation of Fuzzy Connectives

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Abstract: In aggregation problems different types of information items can occur. In spite of such diversity, it is possible to reinterpret them approximately in a unique formal setting by means of profiles: an extension of fuzzy set membership functions. Then the original aggregation problem can be modelled by an appropriate profile aggregation. Therefore, in this paper we concentrate on and reconsider some aggregation operators used especially in fuzzy logic. This view is based on the specific role that the same range – the closed unit interval – plays according to different interpretations. On one hand, triangular norms are capable of reflecting the closed unit interval as a negative scale, while triangular conorms interpret it as a positive scale. Both scales are unipolar. On the other hand, uninorms and nullnorms represent a bipolar scale interpretation of the same closed unit interval.

Key-Words: unipolar and bipolar scales, t-norms, t-conorms, uninorms, nullnorms

1 Introduction

Studying aggregation operations has always been an active field of research. The reason is very simple: aggregation of several inputs into a single output is an indispensable step in diverse procedures of mathematics, physics, engineering, economical, social and other sciences. Mathematical investigations have especially been motivated by the development of fuzzy set theory and fuzzy logics [21], but also by some non-classical approaches to decisions (e.g. multi-criteria decision theory).

There are some well-known traditional aggregation functions, such as the weighted arithmetic mean, which play a key role in probability and other classical areas. Other types of operations consist of logical connectives originated from many-valued logics, with typical examples of triangular norms and conorms [17]. Recently several other classes of aggregation operations have been introduced and studied, such as uninorms [20] and nullnorms [3].

In the present paper we concentrate on specific classes of aggregation operations, which can be considered as basic logical connectives in fuzzy set theory. At first this may seem to be a restrictive consideration because there are many problem classes in which the nature of aggregation requires operations different from fuzzy connectives. An idea of Dubois and Prade [9] makes it possible to reinterpret diverse problems in a unique formal setting by means of a profile, which is a function $\mu$ from a set $X$ (of possible worlds, of states, of alternatives) to a partially ordered set $L$ acting as a scale. This can be called the membership function of a fuzzy set, even if the original definition [21] is much more restrictive. After briefly recalling this approach, we touch upon the importance of scale types. Then we summarize main classes of traditional connectives together with their properties and related results. Some new classes of associative and commutative connectives are also considered. In these cases we emphasize that the same scale – the closed unit interval $[0, 1]$ – is used, but its interpretation can be different.

2 A Unified Approximate Interpretation of Information Items

Dubois and Prade introduced an interesting idea of profiles in [9]. Following Bloch and Hunter [1], they distinguish several kinds of information items according to their intended meaning:

- Observations: They reflect how the world is believed to be in a particular situation of interest.
- Knowledge: As opposed to observations that pertain to particular situations, knowledge means information that describes how the world is in general.
- Preference: Preference information consists of subjective descriptions of an individual or a group of people’s desires about how they would like the world to be.
3 Scale types

The most typical scale in fuzzy set theory is the closed unit interval \( L = [0, 1] \), as the range of membership functions. But \([0, 1]\) is also representative of uncertainty degrees for probability measures and other non classical measures.

In the general case \( L \) is at least partially ordered by some irreflexive and transitive relation \(<\), see [9]. Moreover, \( L \) generally contains a top element denoted by 1, and a bottom denoted by 0. For all other elements \( \ell \in L \) (i.e., \( \ell \neq 0 \) and \( \ell \neq 1 \)) we have \( 0 < \ell < 1 \). There may also exist a third marked element \( \nu \in L \), somewhere “in the middle”. According to different interpretations of these distinguished elements of the scale \( L \), we can have three main types of scales.

- **Positive scales**: If the top element 1 rates the best possible situation, and the bottom 0 has a neutral meaning. \( L \) is called a negative scale if the bottom element rates the worst possible situation and the top has a neutral flavour. Midpoints \( \nu \) play no specific role in these cases. Both scales are considered unipolar.
- **Bipolar scales**: If \( n : L \to L \) is an order-reversing mapping then \( n \) turns a positive scale into a negative one and vice versa.
- **Bipolar scales**: \( L \) is a bipolar scale if the top value rates the best possible situation, and the bottom rates the worst possible one. Then the midpoint \( \nu \) is a kind of neutral element separating positive grades from negative ones.

It is also known from psychological studies that human beings handle the above types of scales. With a little bit different categorization three kinds of scales have been identified [15]:

- **Bounded unipolar scales**, typically \([0, 1]\). This is suitable for e.g. membership degrees, uncertainty, which are bounded notions.
- **Unipolar scales**, not necessarily bounded, typically \( \mathbb{R}^+ \), suitable e.g. for priority degrees (one can always imagine something more prioritary).
- **Bipolar scales** (bounded or unbounded), typically \( \mathbb{R} \), which are suitable for all paired concepts of natural language, such as attraction/repulsion, good/bad, etc.

Now we turn to the handling of some associative operations on negative, on positive, and on bipolar scales.

4 Unipolar Scale: Triangular Norms and Conorms

The original fuzzy set theory was formulated in terms of Zadeh’s standard operations of intersection, union and complement. The axiomatic skeleton used for characterizing fuzzy intersection and fuzzy union are known as triangular norms (t-norms) and triangular conorms (t-conorms), respectively. For more details we refer to the books [10] and [17]. For tunable parametric families we refer to [8].

**Definition 1.** A triangular norm (shortly: a t-norm) is a function \( T : [0, 1]^2 \to [0, 1] \) which is associative, increasing and commutative, and satisfies the boundary condition \( T(1, x) = x \) for all \( x \in [0, 1] \).

For a triangular norm, the top element of the scale is neutral, and the bottom element has a negative meaning. Therefore, \([0, 1]\) is a negative scale now.

**Definition 2.** A triangular conorm (shortly: a t-conorm) is an associative, commutative, increasing \( S : [0, 1]^2 \to [0, 1] \) function, with boundary condition \( S(0, x) = x \) for all \( x \in [0, 1] \).

For a triangular conorm, the top element of the scale is of positive nature, and the bottom element is neutral. Therefore, \([0, 1]\) is a positive scale now.
Notice that continuity of a t-norm and a t-conorm is not taken for granted.

The following are the four basic t-norms, namely, the minimum $T_M$ the product $T_P$, the Łukasiewicz t-norm $T_L$, and the drastic product $T_D$, which are given by, respectively:

$$
T_M(x, y) = \min(x, y),
$$
$$
T_P(x, y) = x \cdot y,
$$
$$
T_L(x, y) = \max(x + y - 1, 0),
$$
$$
T_D(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in [0, 1]^2, \\
\min(x, y) & \text{otherwise}.
\end{cases}
$$

These four basic t-norms have some remarkable properties. The drastic product $T_D$ and the minimum $T_M$ are the smallest and the largest t-norm, respectively. The minimum $T_M$ is the only t-norm where each $x \in [0, 1]$ is an idempotent element. The product $T_P$ and the Łukasiewicz t-norm $T_L$ are prototypical examples of two important subclasses of t-norms (of strict and nilpotent t-norms, respectively).

**Definition 3.** A non-increasing function $N : [0, 1] \to [0, 1]$ satisfying $N(0) = 1, N(1) = 0$ is called a negation. A negation $N$ is called strict if $N$ is strictly decreasing and continuous. A strict negation $N$ is said to be a strong negation if $N$ is also involutive: $N(N(x)) = x$ for all $x \in [0, 1]$. The standard negation is simply $N_s(x) = 1 - x, \quad x \in [0, 1]$. Clearly, this negation is strong. It plays a key role in the representation of strong negations.

We call a continuous, strictly increasing function $\varphi : [0, 1] \to [0, 1]$ with $\varphi(0) = 0, \varphi(1) = 1$ an automorphism of the unit interval.

Note that $N : [0, 1] \to [0, 1]$ is a strong negation if and only if there is an automorphism $\varphi$ of the unit interval such that for all $x \in [0, 1]$ we have

$$
N(x) = \varphi^{-1}(N_s(\varphi(x))).
$$

In what follows we assume that $T$ is a t-norm, $S$ is a t-conorm and $N$ is a strong negation.

Clearly, for every t-norm $T$ and strong negation $N$, the operation $S$ defined by

$$
S(x, y) = N(T(N(x), N(y))), \quad x, y \in [0, 1] \tag{1}
$$

is a t-conorm. In addition, $T(x, y) = N(S(N(x), N(y))) \quad (x, y \in [0, 1])$. In this case $S$ and $T$ are called $N$-duals. In case of the standard negation (i.e., when $N(x) = 1 - x$ for $x \in [0, 1]$) we simply speak about duals. Obviously, equality (1) expresses the De Morgan’s law in the fuzzy case.

Generally, for any t-norm $T$ and t-conorm $S$ we have

$$
T_D(x, y) \leq T(x, y) \leq T_M(x, y)
$$

and

$$
S_M(x, y) \leq S(x, y) \leq S_D(x, y),
$$

where $S_M$ is the dual of $T_M$, and $S_D$ is the dual of $T_D$.

These inequalities are important from practical point of view as they establish the boundaries of the possible range of mappings $T$ and $S$.

Between the four basic t-norms we have these strict inequalities:

$$
T_D < T_P < T_L < T_M.
$$

## 5 Bipolar Scale: Uninorms and Null-norms

### 5.1 Uninorms

Uninorms were introduced by Yager and Rybalov [20] as a generalization of t-norms and t-conorms. For uninorms, the neutral element is not forced to be either 0 or 1, but can be any value in the unit interval.

**Definition 4.** [20] A uninorm $U$ is a commutative, associative and increasing binary operator with a neutral element $e \in [0, 1], i.e., for all $x \in [0, 1]$ we have $U(x,e) = x$.

T-norms do not allow low values to be compensated by high values, while t-conorms do not allow high values to be compensated by low values. Uninorms may allow values separated by their neutral element to be aggregated in a compensating way. The structure of uninorms was studied by Fodor et al. [11]. For a uninorm $U$ with neutral element $e \in [0, 1]$, the binary operator $T_U$ defined by

$$
T_U(x, y) = \frac{U(e, x, e, y)}{e}
$$

is a t-norm; for a uninorm $U$ with neutral element $e \in [0, 1]$, the binary operator $S_U$ defined by

$$
S_U(x, y) = \frac{U(e + (1 - e)x, e + (1 - e)y) - e}{1 - e}
$$

is a t-conorm. The structure of a uninorm with neutral element $e \in [0, 1]$ on the squares $[0, e]^2$ and $[e, 1]^2$ is therefore closely related to t-norms and t-conorms. For $e \in [0, 1]$, we denote by $\phi_e$ and $\psi_e$ the linear transformations defined by $\phi_e(x) = \frac{x}{e}$ and $\psi_e(x) = \frac{e - x}{e}$. To any uninorm $U$ with neutral element $e \in [0, 1]$, there corresponds a t-norm $T$ and a t-conorm $S$ such that:
(i) \( U(x, y) = \phi_e^{-1}(T(\phi_e(x), \phi_e(y))) \) for any 
\((x, y) \in [0, e]^2\),

(ii) \( U(x, y) = \psi_e^{-1}(S(\psi_e(x), \psi_e(y))) \) for any 
\((x, y) \in [e, 1]^2\).

On the remaining part of the unit square, i.e. on \( E = [0, e[ \times e, 1] \cup [e, 1] \times [0, e[ \) it satisfies
\[
\min(x, y) \leq U(x, y) \leq \max(x, y),
\]
and could therefore partially show a compensating behaviour, i.e. take values strictly between minimum and maximum. Note that any uninorm \( U \) is either conjunctive, i.e. \( U(0, 1) = U(1, 0) = 0 \), or disjunctive, i.e. \( U(0, 1) = U(1, 0) = 1 \).

From the point of view of bipolar scales, the unit interval is viewed as the union of two unipolar scales: the interval \([0, e[\) with the underlying t-norm is a negative scale, while the interval \([e, 1]\) with the underlying t-conorm is a positive scale. Therefore, the separating element \( e \) is indeed a neutral one.

5.1.1 Representation of Uninorms

In analogy to the representation of continuous Archimedean t-norms and t-conorms in terms of additive generators, Fodor et al. [11] have investigated the existence of uninorms with a similar representation in terms of a single-variable function. This search leads back to Dombi’s class of aggregative operators [7]. This work is also closely related to that of Klement et al. on associative compensatory operators [16]. Consider \( e \in [0, 1[ \) and a strictly increasing continuous \([0, 1[ \rightarrow \mathbb{R} \) mapping \( h \) with \( h(0) = -\infty \), \( h(e) = 0 \) and \( h(1) = +\infty \). The binary operator \( U \) defined by
\[
U(x, y) = h^{-1}(h(x) + h(y))
\]
for any \( (x, y) \in [0, 1[^2 \setminus \{(0, 1), (1, 0)\} \), and either \( U(0, 1) = U(1, 0) = 0 \) or \( U(0, 1) = U(1, 0) = 1 \), is a uninorm with neutral element \( e \). The class of uninorms that can be constructed in this way has been characterized [11].

Consider a uninorm \( U \) with neutral element \( e \in [0, 1[ \), then there exists a strictly increasing continuous \([0, 1[ \rightarrow \mathbb{R} \) mapping \( h \) with \( h(0) = -\infty \), \( h(e) = 0 \) and \( h(1) = +\infty \) such that
\[
U(x, y) = h^{-1}(h(x) + h(y))
\]
for any \( (x, y) \in [0, 1[^2 \setminus \{(0, 1), (1, 0)\} \) if and only if
(i) \( U \) is strictly increasing and continuous on \([0, 1[^2 \);
(ii) there exists an involutive negator \( N \) with fixpoint \( e \) such that
\[
U(x, y) = N(U(N(x), N(y)))
\]
for any \( (x, y) \in [0, 1[^2 \setminus \{(0, 1), (1, 0)\} \).

The uninorms characterized above are called representable uninorms. The mapping \( h \) is called an additive generator of \( U \). The involutive negator corresponding to a representable uninorm \( U \) with additive generator \( h \), as mentioned in condition (ii) above, is denoted \( N_U \) and is given by
\[
N_U(x) = h^{-1}(-h(x)).
\]

Clearly, any representable uninorm comes in a conjunctive and a disjunctive version, i.e. there always exist two representable uninorms that only differ in the points \((0, 1)\) and \((1, 0)\). Representable uninorms are almost continuous, i.e. continuous except in \((0, 1)\) and \((1, 0)\), and Archimedean, in the sense that \((\forall x \in [0, e[)(U(x, x) < x)\) and \((\forall x \in [e, 1[)(U(x, x) > x)\). Clearly, representable uninorms are not idempotent. The classes \( U_{\min} \) and \( U_{\max} \) do not contain representable uninorms. A very important fact is that the underlying t-norm and t-conorm of a representable uninorm must be strict and cannot be nilpotent. Moreover, given a strict t-norm \( T \) with decreasing additive generator \( f \) and a strict t-conorm \( S \) with increasing additive generator \( g \), we can always construct a representable uninorm \( U \) with desired neutral element \( e \in [0, 1[ \) that has \( T \) and \( S \) as underlying t-norm and t-conorm. It suffices to consider as additive generator the mapping \( h \) defined by
\[
h(x) = \begin{cases} 
-f \left( \frac{x}{e} \right) , & \text{if } x \leq e \\
g \left( \frac{x-e}{1-e} \right) , & \text{if } x \geq e .
\end{cases}
\]

On the other hand, the following property indicates that representable uninorms are in some sense also generalizations of nilpotent t-norms and nilpotent t-conorms: \((\forall x \in [0, 1[)(U(x, N_U(x)) = N_U(e))\). This claim is further supported by studying the residual operators of representable uninorms in [6].

As an example of the representable case, consider the additive generator \( h \) defined by
\[
h(x) = \log \frac{x}{1-x},
\]
then the corresponding conjunctive representable uninorm \( U \) is given by \( U(x, y) = 0 \) if \( (x, y) \in \{(1, 0), (0, 1)\} \), and
\[
U(x, y) = \frac{xy}{(1-x)(1-y) + xy}
\]
otherwise, and has as neutral element \( \frac{1}{2} \). Note that \( N_U \) is the standard negator: \( N_U(x) = 1 - x \).

The class of representable uninorms contains famous operators, such as the functions for combining certainty factors in the expert systems MYCIN and PROSPECTOR [5]. The MYCIN expert system was one of the first systems capable of reasoning under
uncertainty [2]. To that end, certainty factors were introduced as numbers in the interval $[-1, 1]$. Essential in the processing of these certainty factors is the modified combining function $C$ proposed by van Melle [2].

The $[-1, 1]^2 \rightarrow [-1, 1]$ mapping $C$ is defined by

$$C(x, y) = \begin{cases} 
  x + y(1 - x) & \text{if } \min(x, y) \geq 0 \\
  x + y(1 + x) & \text{if } \max(x, y) \leq 0 \\
  1 - \min(|x|, |y|) & \text{otherwise}
\end{cases}$$

The definition of $C$ is not clear in the points $(-1, 1)$ and $(1, -1)$, though it is understood that $C(-1, 1) = C(1, -1) = -1$. Rescaling the function $C$ to a binary operator on $[0, 1]$, we obtain a representable uninorm with neutral element $\frac{1}{2}$ and as underlying t-norm and t-conorm the product and the probabilistic sum.

### 5.2 Nullnorms

**Definition 5.** [3] A nullnorm $V$ is a commutative, associative and increasing binary operator with an absorbing element $a \in [0, 1]$, i.e. $(\forall x \in [0, 1])(V(x, a) = a)$, and that satisfies

\[
\begin{align*}
(\forall x \in [0, a])(V(x, 0) = x) \quad & (4) \\
(\forall x \in [a, 1])(V(x, 1) = x) \quad & (5)
\end{align*}
\]

The absorbing element $a$ corresponding to a nullnorm $V$ is clearly unique. By definition, the case $a = 0$ leads back to t-norms, while the case $a = 1$ leads back to t-conorms. In the following proposition, we show that the structure of a nullnorm is similar to that of a uninorm. In particular, it can be shown that it is built up from a t-norm, a t-conorm and the absorbing element $[3]$.

**Theorem 1.** Consider $a \in [0, 1]$. A binary operator $V$ is a nullnorm with absorbing element $a$ if and only if

(i) if $a = 0$: $V$ is a t-norm;

(ii) if $0 < a < 1$: there exists a t-norm $T_V$ and a t-conorm $S_V$ such that $V(x, y)$ is given by

$$\left\{ \begin{array}{ll}
  \phi^{-1}_a(S_V(\phi_a(x), \phi_a(y))) & \text{if } (x, y) \in [0, a]^2 \\
  \psi^{-1}_a(T_V(\psi_a(x), \psi_a(y))) & \text{if } (x, y) \in [a, 1]^2 \\
  a & \text{otherwise}
\end{array} \right. \quad (6)$$

(iii) if $a = 1$: $V$ is a t-conorm.

From the point of view of bipolar scales, now the unit interval is viewed as the union of two unipolar scales: the interval $[0, a]$ with the underlying t-conorm is a positive scale, while the interval $[a, 1]$ with the underlying t-norm is a negative scale. This is just the opposite to the case of uninorms.

Recall that for any t-norm $T$ and t-conorm $S$ it holds that $T(x, y) \leq \min(x, y) \leq \max(x, y) \leq S(x, y)$, for any $(x, y) \in [0, 1]^2$. Hence, for a nullnorm $V$ with absorbing element $a$ it holds that $(\forall (x, y) \in [0, a]^2) (V(x, y) \geq \max(x, y))$ and $(\forall (x, y) \in [a, 1]^2) (V(x, y) \leq \min(x, y))$. Clearly, for any nullnorm $V$ with absorbing element $a$ it holds for all $x \in [0, 1]$ that

$$V(x, 0) = \min(x, a) \quad \text{and} \quad V(x, 1) = \max(x, a). \quad (7)$$

Notice that, without the additional conditions (4) and (5), it cannot be shown that a commutative, associative and increasing binary operator $V$ with absorbing element $a$ behaves as a t-conorm and t-norm on the squares $[0, a]^2$ and $[a, 1]^2$.

Nullnorms are a generalization of the well-known median studied by Fung and Fu [13], which corresponds to the case $T = \min$ and $S = \max$. For a more general treatment of this operator, we refer to [14]. We recall here the characterization of that median as given by Czogała and Drewniak [4]. Firstly, they observe that an idempotent, associative and increasing binary operator $O$ has as absorbing element $a \in [0, 1]$ if and only if $O(0, 1) = O(1, 0) = a$. Then the following theorem can be proven.

**Theorem 2.** [4] Consider $a \in [0, 1]$. A continuous, idempotent, associative and increasing binary operator $O$ satisfies $O(0, 1) = O(1, 0) = a$ if and only if it is given by

$$O(x, y) = \begin{cases} 
  \max(x, y) & \text{if } (x, y) \in [0, a]^2 \\
  \min(x, y) & \text{if } (x, y) \in [a, 1]^2 \\
  a & \text{elsewhere}
\end{cases}$$

Nullnorms are also a special case of the class of $T$-$S$ aggregation functions introduced and studied by Fodor and Calvo [12].

**Definition 6.** Consider a continuous t-norm $T$ and a continuous t-conorm $S$. A binary operator $F$ is called a $T$-$S$ aggregation function if it is increasing and commutative, and satisfies the boundary conditions

$$\begin{align*}
(\forall x \in [0, 1])(F(x, 0) = T(F(1, 0), x)) \\
(\forall x \in [0, 1])(F(x, 1) = S(F(1, 0), x)).
\end{align*}$$

When $T$ is the algebraic product and $S$ is the probabilistic sum, we recover the class of aggregation functions studied by Mayor and Trillas [18]. Rephrasing a result of Fodor and Calvo, we can state that the
class of associative $T$-$S$ aggregation functions coincides with the class of nullnorms with underlying t-norm $T$ and t-conorm $S$.

Note finally that so-called pseudo-analysis intensively uses operations similar to uninorms and nullnorms, but on other scales, see for example [19].

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