Quintic B-spline Curve Generation and Modification based on specified Radius of Curvature

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Abstract: - A method to generate a quintic B-spline curve which passes through the given points is described. In this case, there are four more unknown control point positions than there are equations. To overcome this problem three methods are described. First, solve the underdetermined system as it stands. Secondly, decrease the number of unknown control point positions in an underdetermined system in order to convert it to a determined system. Third, a method to increase the number of equations is employed to change an underdetermined system to a determined system. In addition to this, another method to generate a quintic B-spline curve using given points with gradients in sequence is described. In this case, a linear system will be either overdetermined, determined or underdetermined. This depends on the number of given points with gradients in sequence. Additionally, a method to modify a quintic B-spline curve is described. The objective is to change an aesthetically unpleasing curve to an aesthetically pleasing curve. This is accomplished by minimizing the difference between the quintic B-spline curves radius of curvature and the specified radius of curvature using the least-squares method.

Key-Words: - B-spline curve generation, curvature vector, curve shape modification, given points, given points with gradients, underdetermined system, overdetermined system

1 Introduction

A NURBS curve, which is commonly used in the field of CAD・CAM and Computer Graphics, is used as an expression of a freeform curve. Particularly, cubic NURBS curves are widely used. In the smoothing of curves, it may be desirable to interpolate second derivative information at the knots. This is not possible with cubic splines, and so splines of higher degree have to be used. For symmetric boundary conditions, it is more convenient to work with quintic splines [1].

In this study, radius of curvature ranging over multi segments of a NURBS curve is modified based on the specified radius of curvature. Therefore, a quintic NURBS curve is needed in this study. In addition to this, the weights of the NURBS curve are set to one. For this reason, the curve used in this study is a quintic B-spline curve.

The objective of this study is to develop a B-spline curve generation method using given points in sequence for practical use.

Positions and gradients are given to the B-spline curve equations and first derivative equations of the B-spline curve respectively. Then, a B-spline curve is generated. Afterwards, if necessary, the shape of this B-spline curve is modified according to the target radius of curvature that has been smoothed.

Using the least-squares method, the shape of the designed curve is modified based on the target radius of curvature specified.

There are many related works for generation of a curve. There are data fitting by interpolation [2, 3], and data fitting with B-spline [4], and data fitting by principal component analysis following the statistical method [5].

There are many related works for generation of a fair curve dealing with knots. These are knot insertion [6, 7], knot removal algorithm for B-spline curves [8], knot removal algorithm for NURBS curves [9], and fair curve generation by knot value and weight modification [10].

There are many related works for generation of fair curvature distribution. Fair curvature distribution algorithms by modifying knot spacing [11, 12], and by removing and reinserting knots [13-17] have been published.

Curve generation algorithms related to curvature by modifying the control points have been published.
These use a clothoidal curve for specifying the curvature [18], and modify the shape of the curve based on the target radius of curvature specified [19-21]. Curve generation algorithms related to curvature by specifying curvature distribution have also been published [22].

Section 2 of this paper describes a quintic B-spline curve, the derivatives of a quintic B-spline curve, curvature vector, curvature, and radius of curvature. Section 3 describes the generation of a quintic B-spline curve which passes through the given points in sequence and the generation of a quintic B-spline curve using the given points with gradients in sequence. In section 4, B-spline curve shape modification based on the specified radius of curvature is described.

2 B-spline Curve Expression

An \( n \) \( n \geq 6 \) segments \( n \geq 6 \) quintic B-spline curve is composed of \( n \) control points such as \( q_0, q_1, \ldots, q_{n-1} \) as

\[
R(t) = \sum_{i=0}^{n} N_{i,6}(t) \cdot q_i, \quad (1)
\]

where \( N_{i,6}(t) \) \( i = 0, 1, \ldots, n-1 \) are B-spline basis functions.

These functions are defined by the de Boor-Cox [23] recursion formulas, and are recursively defined by knot sequence \( t_0, t_1, t_2, \ldots, t_{n+3} \) as

\[
N_{i,6}(t) = \begin{cases} 
1 & (i_i \leq t < i_{i+j}) \\
0 & \text{otherwise} 
\end{cases}, \quad (2)
\]

\[
N_{i,j}(t) = \frac{t-t_i}{t_{i+j}-t_i} N_{j,6}(t) + \frac{t_{i+j}-t}{t_{i+j}-t_{i+1}} N_{j,6}(t),
\]

where \( j = 0, 1, \ldots, n+4 \) and \( M = 2, 3, \ldots, 6 \), \( j \) corresponds to the knot number.

If the knot vector contains repeated knot values called multiple knots, then a division of the form \( N_i(t/) = (t_i, t_{i+1} \ldots t_j) = 0/0 \) (for some \( j \) may be encountered during the execution of the recursion. Whenever this occurs, it is assumed that \( 0/0 = 0 \) [24].

A one segment quintic B-spline curve with the knot vector \[ -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6 \] is

\[
R(t) = \frac{1}{120} \begin{cases} 
(1-t)^5 q_0 \\
+ (5t^5 - 20t^4 + 20t^2 - 8)q_1 \\
+ (-10t^4 + 30t^3 - 60t^2 + 26)q_2 \\
+ (10t^3 - 20t^2 + 20t^2 + 50t + 26)q_3 \\
+ (-5t^4 + 5t^3 + 10t^2 + 5t + 10)q_4 \\
+ t^4 q_5 \end{cases}. \quad (3)
\]

The first derivative of a quintic B-spline curve shown in Eq.(3) is expressed as

\[
\frac{dR(t)}{dt} = \frac{1}{24} \begin{cases} 
(1-t)^4 q_0 \\
+ (5t^4 - 20t^3 + 20t^2 + 8t - 10)q_1 \\
+ (-10t^3 + 24t^2 - 24)q_2 \\
+ (10t^2 - 16t^1 + 12t^2 + 8t + 10)q_3 \\
+ (-5t^3 + 4t^2 + 6t^2 + 4t + 1)q_4 \\
+ t^3 q_5 \end{cases}. \quad (4)
\]

The second derivative of a quintic B-spline curve shown in Eq.(3) is expressed as

\[
\frac{d^2 R(t)}{dt^2} = \frac{1}{6} \begin{cases} 
(1-t)^3 q_0 \\
+ (5t^3 - 12t^2 + 6t + 2)q_1 \\
+ (-10t^2 + 18t - 6)q_2 \\
+ (10t^2 - 12t^2 - 6t + 2)q_3 \\
+ (-5t^3 + 3t^2 + 3t + 1)q_4 \\
+ t^2 q_5 \end{cases}. \quad (5)
\]

The third derivative of a quintic B-spline curve shown in Eq.(3) is expressed as

\[
\frac{d^3 R(t)}{dt^3} = \frac{1}{2} \begin{cases} 
(1-t)^2 q_0 \\
+ (5t^2 - 8t - 2)q_1 \\
+ (-10t^2 + 12t)q_2 \\
+ (10t^2 - 8t - 2)q_3 \\
+ (-5t^2 + 2t + 1)q_4 \\
+ t q_5 \end{cases}. \quad (6)
\]

The fourth derivative of a quintic B-spline curve shown in Eq.(3) is expressed as

\[
\frac{d^4 R(t)}{dt^4} = (1-t)q_0 \\
+ (5t - 4)t q_1 \\
+ (-10t + 6)t q_2 \\
+ (10t - 4)q_3 \\
+ (-5t + 1)q_4 \\
+ tq_5. \quad (7)
\]

Curvature vector is expressed as

\[
\kappa(t) = \frac{R(t) \times \hat{R}(t) \times R(t)}{|R(t)|^5}, \quad (8)
\]

where \( R(t) \) is the first derivative of a B-spline curve, and \( \hat{R}(t) \) is the second derivative of a B-spline curve.

Curvature is the magnitude of the curvature vector, therefore, curvature is expressed as

\[
\kappa(t) = |\kappa(t)|. \quad (9)
\]
By definition, the curvature of a curve is nonnegative. However, in many cases it is useful to ascribe a sign to the curvature [25]. The choosing of the sign is commonly connected with the tangent rotation by moving along the curve in the direction of the increasing parameter. The curvature of the curve is positive when its tangent rotates counter-clockwise, the curvature of the curve is negative when its tangent rotates clockwise.

The radius of curvature is the reciprocal number of curvature, therefore, the radius of curvature is expressed as

$$\rho(i) = \frac{1}{\kappa(i)}.$$  \hspace{1cm} (10)

3 Generation of a B-spline Curve

In this section, methods to generate a quintic B-spline curve which passes through the given points in sequence and to generate a quintic B-spline curve using the given points with gradients in sequence are described.

One widely used form of data fitting is interpolation. Sometimes the problem is to interpolate to positional data alone, and sometimes it is necessary to interpolate to derivatives as well as positions, at least at some of the points [26].

3.1 Generation of a Quintic B-spline Curve which Passes through Given Points in Sequence

A method to generate a quintic B-spline curve which passes through the given points in sequence is described.

To simplify further description, notations shown in equations hereafter are summarized. \(i\) is the ordinal number of the given points in sequence, \(m\) serves as both the last ordinal number of the given points in sequence and the total number of B-spline curve segments, \(m+1\) is the total number of given points, and \(m+5\) is the total number of control points. \(j\) is the ordinal number of the given gradients assigned to the given points, \(n\) is the last ordinal number of the given gradients, and \(n+1\) is the total number of given gradients.

The parameter of Eq.(3) is set to zero by defining the geometrical knot position corresponding to the knot of the knot vector. Then, Eq.(3) is expressed as

$$R_i = \frac{1}{120}\begin{bmatrix} q_i + 26q_{i+1} + 66q_{i+2} + 26q_{i+3} + q_{i+4} \end{bmatrix},$$ \hspace{1cm} (11)

\((i = 0, 1, 2, 3, \cdots, m)\)

The positional vectors \(P_i(i = 0, 1, 2, 3, \cdots, m)\) of the given points in sequence are assigned to \(R_i(i = 0, 1, 2, 3, \cdots, m)\) in Eq.(11), and \(q_0, q_1, q_2, q_3, q_4 (i = 0, 1, 2, 3, \cdots, m)\) are the control points of a quintic B-spline curve.

When the control points of a quintic B-spline curve are calculated using Eq.(11), the number of unknowns, which are the positions of the control points, are four more than the number of equations which are expressed by Eq.(11). That is, this linear system to be solved to obtain the control points is underdetermined. Three methods are used as follows. One, a method to calculate the control point positions in an underdetermined system as it stands. Two, a method to convert an underdetermined system to a determined one by expressing the unknown control points by the linear combination of known control points. Three, convert an underdetermined system to a determined system by assigning four gradients to given points to increase the number of equations.

For an underdetermined system, while setting auxiliary function, the linear system is solved under the constraint condition by selecting one solution from an infinite number of exact solutions using Lagrange's method of indeterminate multipliers. In another method, the curvatures at both ends of the quintic B-spline curve are assumed to be zero. Then, the second derivatives at both ends of the quintic B-spline curve are set to zero, while setting the parameter of Eq.(3) at zero by defining the geometrical knot position corresponding to the knot of the knot vector. The following equations are then obtained.

$$q_0 + 2q_1 - 6q_2 + 2q_3 + q_4 = \theta \hspace{1cm} (12)$$

$$q_n - 4q_{n+1} + 6q_{n+2} - 4q_{n+3} + q_{n+4} = \theta \hspace{1cm} (13)$$

Accordingly, the fourth derivatives at both ends of the quintic B-spline curve are set to zero. Then, in the same manner, the following equations are obtained.

$$q_0 - 6q_1 + 12q_2 - 6q_3 + q_4 = \theta \hspace{1cm} (12)$$

$$q_n - 4q_{n+1} + 6q_{n+2} - 4q_{n+3} + q_{n+4} = \theta \hspace{1cm} (13)$$

Using Eq.(12) and Eq.(13), Eq.(14) is obtained.

$$q_0 = 2q_1 - q_2$$

$$q_1 = 2q_2 - q_3$$

$$q_{n+1} = 2q_{n+2} - q_{n+3}$$

$$q_{n+4} = 2q_{n+5} - q_n$$ \hspace{1cm} (14)

In Eq.(14), unknown control points \(q_0, q_1, q_{n+3},\) and \(q_{n+4}\) are expressed by the linear combination of known
control points. Thus, \( q_0, q_1, q_{n+1} \), and \( q_{n+4} \) are made known. Therefore, the number of equations will be equal to the number of unknowns. That is, this linear system is determined.

\[ P_0, P_1, \ldots, P_{n+1}, P_n \] are given points. \( q_0, q_1, q_{n+1}, q_{n+4} \) are control points of a quintic B-spline curve.

Using different boundary derivatives has a strong influence on the shape and approximation quality of the curve [27, 28].

A quintic B-spline curve which passes through the given points in sequence, assuming the curvatures at both ends of the quintic B-spline curve are zero, and \( q_0, q_1, q_{n+1}, q_{n+4} \) in Eq.(14) are illustrated in Fig.1.

\[ P_i (i = 0, 1, 2, 3, \ldots, m) \] in Fig.1 are the positional vectors of the given points in sequence, and are assigned to \( R_i (i = 0, 1, 2, 3, \ldots, m) \) in Eq.(11). As shown in Fig.1, both ends of a quintic B-spline \( P_0 \) and \( P_m \), which are also the given points, have the same position of the control point \( q_0 \) and \( q_{n+2} \). This can be derived from Eq.(3) and Eq.(14). \( P_1 \) is the midpoint of \( q_0 \) and \( q_1 \). Also, it is midpoint of \( q_0 \) and \( q_2 \). Another end point \( P_m \) is the midpoint of \( q_{n+1} \) and \( q_{n+4} \). Also, it is the midpoint of \( q_n \) and \( q_{n+1} \). This can be derived from Eq.(14).

As another method, in addition to the given points in sequence, gradients are assigned to the given points to compensate for the difference between the number of unknowns and that of the equations. For this, the location of the four gradients is assigned situationally.

Eq.(15) is applied to the gradients \( d_j (j = 0, 1, 2, 3, \ldots, n) \) by setting the parameter of Eq.(4) to zero by defining the geometrical knot position corresponding to the knot of the knot vector.

\[
\frac{dR}{dt} = \frac{1}{24} (-q_j - 10q_{j+1} + 10q_{j+3} + q_{j+4}) ,
\]

\[(j = 0, 1, 2, 3, \ldots, n)\]

The \( j \) shown in Eq.(15) corresponds to the \( i \) in Eq.(11) and is assigned to the given points situationally.

The defined gradients are located at the beginning given point and it’s adjacent point, and at the end given point and its adjacent point in general. As a magnitude of the first derivative, the value of the distance between the two adjacent given points is assigned as a default value. As for further adjustment, the magnitude of the first derivative is determined interactively [29].

Using the given points in sequence and four location specified gradients, the linear system becomes determined. That is, the number of unknowns is equal to the number of equations.

The concept of a quintic B-spline curve generation using the given points in sequence and four location specified gradients is illustrated in Fig.2. The defined gradients are located at the beginning given point and its adjacent point, and at the end given point and its adjacent point. That is, the \( j \) of Eq.(15) are determined as 0, 1, \( n-1 \), and \( n \) respectively. \( P_0, P_1, P_2, P_3, P_{n+1}, P_{n+2}, P_{n+3}, \) and \( P_n \) are the positional vectors of the given points in sequence, and \( d_0, d_1, d_{n-1}, \) and \( d_n \) are the four location specified gradients.

As examples of a quintic B-spline curve which passes through the given points in sequence, three quintic B-spline curves which simulate filleting curves are shown with their curvature plots in Fig.3. In Fig.3(a), and (b), the \( j \) of Eq.(15) are determined situationally as 0, 2, 4, and 6. In Fig.3(c), the \( j \) of Eq.(15) are determined situationally as 0, 1, 3, and 4. In the case of a curve including a filleting segment, note that two gradients are placed at the start and end points of the filleting segment as well as the quintic B-spline curve beginning and end points.
3.2 Generation of a Quintic B-spline Curve using Given Points with Gradients

In this sub-section, a quintic B-spline curve generation using given points with gradients in sequence is described. The concept of generating a quintic B-spline curve using given points with gradients in sequence is illustrated in Fig.4.

A quintic B-spline curve is generated by solving Eq.(11) and Eq.(15) simultaneously by making \( m \) in Eq.(11) equal to \( n \) in Eq.(15). In this case, the \( i \) in Eq.(11) corresponds to the \( j \) in Eq.(15). If the number of given points with gradients is 4, the number of B-spline curve equations (Eq.(11)) is 4 and the number of first derivative equations (Eq.(15)) is 4. As a linear system, the total number of equations is 8, whereas the total number of control points of a quintic B-spline curve is 8. Therefore, this linear system is determined. That is, the rank of a coefficient matrix of a linear system is equal to the number of unknowns. The solution to this linear system is exact.

But, in case the number of given points with gradients is 3, the number of equations (Eq.(11)) which pass through the given points is 3, and the number of equations of the first derivative (Eq.(15)) is 3. In this case, as a linear system, the total number of equations is 6, whereas the number of control points of the quintic B-spline curve is 7. That is, the number of equations is less than the number of unknowns. Therefore, this linear system is underdetermined [30].

For an underdetermined system, while setting auxiliary function, the linear system is solved under the constraint condition by selecting one solution from an infinite number of exact solutions using Lagrange's method of indeterminate multipliers.

In case the number of given points with gradients is 5, the number of equations (Eq.(11)) is 5, and the number of equations of the first derivative (Eq.(15)) is 5. In this case, as a linear system, the total number of equations is 10, whereas the number of control points of the quintic B-spline curve is 9. That is, the number of equations exceeds the number of unknowns. Therefore, this linear system is overdetermined [31].

For an overdetermined system, the differences between the right and left sides of all the equations of the system are minimized. The control points calculated are therefore an approximation.

For a system where the number of given points with gradients is 5 or more, the linear system is overdetermined. For these systems, in accordance with the increments of the differences between the number of equations and the number of unknowns, the status of the approximation worsens.

The above mentioned situations are summarized in Table 1. A determined linear system is shown by the cross hatching.

![Fig.4 Concept of generating a quintic B-spline curve using given points with gradients in sequence](image_url)

<table>
<thead>
<tr>
<th>(I) number of given points with gradients</th>
<th>(II) number of control points of a quintic B-spline curve</th>
<th>(III) system condition (underdetermined, determined, overdetermined)</th>
<th>(IV) solution status</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
<td>underdetermined</td>
<td>exact</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>underdetermined</td>
<td>exact</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>determined</td>
<td>exact</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>overdetermined</td>
<td>approximation</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>overdetermined</td>
<td>approximation</td>
</tr>
<tr>
<td>7</td>
<td>11</td>
<td>overdetermined</td>
<td>approximation</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td>overdetermined</td>
<td>approximation</td>
</tr>
</tbody>
</table>

a linear system is equal to the number of unknowns. The solution to this linear system is exact.

![Fig.3 Examples of a quintic B-spline curve which passes through given points in sequence and four location specified gradients](image_url)
4 Curve Shape Modification based on Smoothed Radius of Curvature Distribution

In this section, a method to modify a quintic B-spline curve shape according to the specified radius of curvature distribution to realize an aesthetically pleasing freeform curve is described.

Radius of curvature is suitable, because it conforms to our visual recognition of the shape of the curve. In a case where curve shape is very close to a straight line, the radius of curvature becomes infinity. Also, at the point of inflection, curvature value becomes zero. Therefore, radius of curvature value becomes infinite. For these reasons, radius of curvature value is converted to curvature value for computation.

The concept of radius of curvature specification and quintic B-spline curve shape modification based on the specified radius of curvature distribution is shown in Fig.5. A quintic B-spline curve and its radius of curvature plots are shown in Fig.5(a).

A method to modify the shape of the quintic B-spline curve shown in Fig.5(a) to the curve shown in Fig.5(b) is examined.

Radius of curvature plots shown in Fig.5(a) are drawn inward from and perpendicular to the curve using straight lines. The length of the line is proportional to the radius of curvature at that spot on the curve. However, the straight lines are not parallel to each other and the beginning points of the individual straight lines are different. Therefore, a curve with a radius of curvature display is suitable to examine the variation of radius of curvature as a whole. But, it is not suitable to examine the length of the straight lines and variation of radius of curvature in detail.

Therefore, considering the parameter of the quintic B-spline curve is different from the perimeter of the curve, the perimeter of a quintic B-spline curve as a straight line is set to the horizontal axis, and the radius of curvature is set to the vertical axis as shown in Fig.5(c). Then, the radius of curvature distribution to the perimeter is drawn. To determine the target radius of curvature distribution by using algebraic functions has been published [19-21].

An example is shown in Fig.5(c). The \( \delta_i \) of radius of curvature distribution of a perimetrically represented quintic B-spline curve is denoted as \( \hat{\rho_i} \), the specified radius of curvature at the same spot is denoted as \( \rho_i \), the difference \( \delta_i \) is shown by Eq.(16) and is illustrated in Fig.5(c).

\[
\delta_i = \rho_i(q_i, \ldots, q_{i+2}, q'_i, \ldots, q'_{i+2}) - \hat{\rho}_i \tag{16}
\]

Where \( i = 0, 1, 2, \ldots, m-1 \), \( m \) is the number of specified radius of curvature, and \( n \) is the number of B-spline curve segments plus 5, which is the degree of the curve.

\[
S(q_i', \ldots, q_{i+2}', q_i, \ldots, q_{i+2}) \quad \text{which is the sum of the squared differences for all specified radius of curvatures in Eq.(17) is minimized by introducing the least-squares method. The radius of curvature expression is non-linear. Therefore, by Taylor's theorem, Eq.(17) is linearized as in Eq.(18).}
\]

\[
S(q_i', \ldots, q_{i+2}', q_i, \ldots, q_{i+2}) \leq \sum_{i=0}^{m-1} (\rho_i(q_i, \ldots, q_{i+2}, q'_i, \ldots, q'_{i+2}) - \hat{\rho}_i)^2 \tag{17}
\]
\[ S(q_i' + \Delta q_i', \cdot, q_{i-2}' + \Delta q_{i-2}', q_i' + \Delta q_i', \cdot, q_{i-1}' + \Delta q_{i-1}') \]
\[ = \sum_{i=1}^{n-1} \rho_i(q_i' + \Delta q_i', q_i', q_{i-1}', q_{i-2}') + \frac{\partial \rho_i}{\partial q_i} \Delta q_i' + \cdot \cdot \cdot \cdot \cdot + \frac{\partial \rho_i}{\partial q_{i-2}} \Delta q_{i-2}' \]  
(18)
\[ \cdot \cdot \cdot + \frac{\partial \rho_i}{\partial q_{i-2}} \Delta q_{i-2}' + \frac{\partial \rho_i}{\partial q_i} \Delta q_i' + \cdot \cdot \cdot + \frac{\partial \rho_i}{\partial q_{i-2}} \Delta q_{i-2}' - \hat{\rho}_i \]  
Eq. (18) is minimized by equating to zero all the partial derivatives of \( S(q_i' + \Delta q_i', \cdot, q_{i-2}' + \Delta q_{i-2}', q_i' + \Delta q_i', \cdot, q_{i-1}' + \Delta q_{i-1}') \) with respect to \( \Delta q_i' \) and \( \Delta q_j' \) \((r = 1,2,\cdot, n-2)\) as
\[ \frac{\partial S}{\partial \Delta q_i'} = 0 \]  
\[ \frac{\partial S}{\partial \Delta q_j'} = 0 \]  
\((r = 1,2,\cdot, n-2)\)
(19)
Using these simultaneous linear equations, \( \Delta q_i' \) and \( \Delta q_j' \) \((r = 1,2,\cdot, n-2)\) are calculated. Then, \( q_i' \) and \( q_j' \) are determined.

This kind of study on the radius of curvature, or the curvature to realize a fair curve is called a constrained non-linear minimization problem [32]. For computation, \( \rho_i \) and \( \hat{\rho}_i \) are calculated based on the perimeter. Then, the perimeter used is converted to the parameter to calculate the position of the control points of the quintic B-spline curve. Thus, a quintic B-spline curve is generated. The total length of the curve, which is the perimeter, is calculated and rescaled as 1. Repeating these operations, positions of the control points of the quintic B-spline curve are determined while \( \delta_i \) \((i = 0,1,\cdot, m - 1)\) are minimized for the entire perimeter.

Smoothed radius of curvature distribution, which is specified as target radius of curvature distribution is shown in Fig. 5(c). Using the above mentioned method, the shape of the curve is modified based on the radius of curvature distribution specified. The dotted line shown in Fig. 5(d) is the smoothed radius of curvature distribution shown in Fig. 5(c). It is visually recognized that the radius of curvature distribution of the shape modified curve shown in Fig. 5(d) matches to the specified radius of curvature.

5 Conclusion
A method to generate a quintic B-spline curve which passes through the given points in sequence is described. In this case, the number of unknowns, which are the positions of the control points, are four more than the number of equations which express quintic B-spline curves. In other words, there are four more unknowns than there are equations.

Two methods have been developed to compensate for the difference between the number of unknowns and that of the equations. These are assuming that the curvatures at both ends of the curve are zero, and assigning four gradients to the given points situationally.

In addition to these methods, another method to generate a quintic B-spline curve which passes through the given points and which has the first derivatives at these given points has been developed. In this case, a linear system will be underdetermined, determined or overdetermined depending on the number of given points with gradients.

For an underdetermined system, the linear system is solved under the constraint condition while setting auxiliary function by selecting one solution from an infinite number of exact solutions using Lagrange’s method of indeterminate multipliers.

For an overdetermined system, the differences between the right and left sides of all the equations of this linear system are minimized.

A method to modify a quintic B-spline curve shape according to the specified radius of curvature distribution to realize an aesthetically pleasing freeform curve is described. The difference between the B-spline curve radius of curvature and the specified radius of curvature is minimized by introducing the least-squares method. A reverse computational technique is applied to solve this problem. This kind of study on the radius of curvature, or the curvature to realize a fair curve is called a constrained non-linear minimization problem.
References:


