Fitting the sigmoid function concerning a Bingham fluid

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Abstract: The exact solution to the equation governing the flow of a Bingham fluid along the cylindrical pipeline may be expressed as the solution of the simplified equation modified by so-called correction term $\epsilon$. The formula $E = 1 - \epsilon$ defines a sigmoid function in the yield number $Y$. In the paper there are presented the approximations to $E$ by a generalized logistic function, by parametrical Bézier representations and by splines. Obtained formulas are examined, also from the point of engineer’s view and it is concluded that a spline fit is the best one discussed.

Key-Words: rheology, mechanics of fluids, best point approximation, numerical methods and their education

1 Introduction

In the mechanics of non-Newtonian fluids there are formulated many models which are essential in the recognition of rheological properties of homogeneous fluids and two-phase systems. A synthetic survey of these models can be found, a.o., in monographs [8], [10] and [14]. Among these models there is the concept developed in 1916 by E.C. Bingham [2], [3] and called a Bingham model. This model is still widely used and investigated. Among the equations governing this model there are rather complicated ones, in particular for a common engineer dealing with the subject.

In this paper there is considered such an expression (appeared recently in the rheological investigations [6]) and there are given much simpler formulas approximating it. There is assumed that reader knows basic notions in the numerical approximation of real functions, such as the least square method, Bernstein polynomials and Bézier curves (one can find them, e.g. in [4], [5]) and the paper is presented in the way to be directly used in the lessons in numerical methods in the intermediate level.

2 Correction term for a Bingham fluid

In 1928 R.O.Herzog and K.Weissenberg developed the equation governing the steady laminar flow of purely viscous fluid in a pipeline. In 1929 B.Rabinowitsch and in 1931 M.Mooney recognized the significance of this equation and now it is called Rabinowitsch-Mooney equation. When applied to any Bingham fluid it turns into the Buckingham-Reiner equation,

\[
A^4 - A \cdot A^3 + \frac{256}{3} \cdot \delta^4 = 0,
\]

where

$A$ is the loss of the pressure in a Bingham fluid related to the unit length, [N/m$^3$],

$A \sim$ is the related loss of pressure calculated from the simplified equation (2), [N/m$^3$],

$\delta = \frac{\tau_0}{D}$, 

$\tau_0$ is the yield stress, [N/m$^2$],

$D$ is the diameter of the pipeline, [m].

Let’s introduce a simplified Buckingham-Reiner equation by the neglecting of the free term in (1); so the simplified equation reads as follows

\[
A^4 - A \cdot A^3 = 0.
\]

In [6] there is derived the exact formula for so-called correction term

\[
\epsilon = \frac{A}{A \sim},
\]

and it is $\epsilon = \epsilon_Y(Y)$, where

\[
\epsilon_Y(Y) = \frac{1}{4} + \frac{1}{4} \cdot \sqrt{\frac{3}{2} \cdot W \cdot 4/3 \cdot \alpha + 1},
\]

\[
\alpha = \frac{1}{8} - \frac{3}{32} \cdot W \cdot 4/3 \cdot \alpha - \frac{1}{16} + \frac{8}{3} \cdot W \cdot 4/3 \cdot \alpha + 1,
\]

\[
\epsilon_Y(Y) = \frac{3}{1 + \sqrt{1 - W^4}} + \frac{3}{1 - \sqrt{1 - W^4}},
\]
The dimensionless quantity $Y$ defined by (7) is called a **yield number** for the Bingham plastic. The yield number was introduced in [7]. The equality

$$Y = \frac{He}{Re},$$

relates the yield number to two other characteristic numbers in the mechanics of fluids: Hedström number $He$ and Reynolds number $Re$, proposed in 1952 and 1883, respectively. The yield number $Y$ assumes values from 0 (for a Newtonian fluid) to the infinity (for a Saint-Venant body), so it’s natural to deal with it the logarithmic scale,

$$x = \log_{10}(Y).$$

In Fig.1 there is shown the graph of the function $E = E(Y)$, where

$$E(Y) = 1 - \varepsilon |Y|.$$

We see that $E$ is a sigmoid function bounded by 0 and 0.25. Although its graph is smooth, in an engineer’s practice (also when using a computer) it is hardly time-consuming to calculate its values. Hence there is a need to have a much simpler expression producing a good approximation to the value $E(Y) = E(10^x)$.

In the next we present such expressions obtained in the class of generalized logistic functions, Bézier curves and splines (including their smooth combinations). In all cases we produce the approximation taking into account arbitrarily chosen pairs abscissas $Y_j$ of points $(Y_j, E_j)$, where $E_j = E(Y_j)$.

### 3 Logistic fitting

In 1838 Pierre-François Verhulst started to elaborate a model of the population growth. He published the results of his investigations in papers *Recherches mathématiques sur la loi d’accroissement de la population* (1845) and *Deuxième mémoire sur la loi d’accroissement de la population* (1847). Since then his model found applications in numerous fields, including demography, biology, chemistry, probability and economics. A crucial element of this model is a logistic function. It is defined by the formula

$$G(x) = \frac{a}{1 + b \cdot e^{-g \cdot x}},$$

where $a$, $b$, $g$ are real parameters, $p$ is arbitrary fixed positive number different from 1. Mostly it is applied with $a > 0$, $b > 1$ and $p = 10$ or Euler number $e$. We can be generalized to have

$$G(x) = \frac{a}{1 + 10^{g \cdot x}},$$

where $a$ and $p$ are as above, $g(x)$ is a polynomial of a certain degree $m$. So, it has $m+2$ parameters and

$$\lim_{x \to \infty} G(x) = a \quad \text{if} \quad \lim_{x \to \infty} g(x) = -\infty.$$

Obviously, to approximate our curve $E = E(Y)$ we put $a = 0.25$ and, having decided to deal with $p = 10$, we look for an appropriate polynomial $w$ of degree as small as possible. First we transform the approximating function to the form

$$g(x) = \log_{10}\left(\frac{0.25}{G(x)} - 1\right).$$

We arbitrarily choose 15 points $(Y_j, E_j)$ laying on the graph of the curve $E = E(Y)$, $x_j = \log_{10}(Y_j) \in \{0.3, 0.5, 0.7, 0.9, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 7, 9\}$. Next we fix the degree $m$. At last, by the least square method

![Fig.1. Graphs $E = E(Y)$ and $E = G(Y)$; there are also shown the points $(Y_j, E_j)$ laying on the graph $E = E(Y)$ and used to produce the graph least square approximation $G(Y)$ to $E(Y)$.](image1)

![Fig.2. Graphs of: a) absolute and b) relative error, i.e., $y = |E(Y) - G(\log_{10}(Y))|$, $y = |E(Y)/G(\log_{10}(Y)) - 1|$](image2)
applied with points \((x_j, G_j)\), where

\[
G_j = \log_{10}\left(\frac{0.25}{E_j} - 1\right),
\]

we get the coefficients of the polynomial \(g\). Results for \(m < 4\) are unsatisfactory, for \(m = 4\) we get

\[
g(x) = 0.004518x^4 - 0.100188x^3 + 0.770748x^2 - 2.94013x + 3.48969.
\]

Taking into account an engineer’s accuracy we examined that, within considered interval \((-1, 9)\), there practically brings no loose in accuracy to have

\[
g(x) = 0.0045x^4 - 0.1x^3 + 0.77x^2 - 2.94x + 3.49.
\]

Graphs \(E = E(Y)\) of the primary function and its approximation

\[
E = G(\log_{10}(Y)) = \frac{0.25}{1 + 10^{0.0045x^4 - 0.1x^3 + 0.77x^2 - 2.94x + 3.49}}
\]

are both shown in Fig.1. In Fig.2 there are graphs of the absolute error and the relative error; obviously, we are not worry the last one is so high where \(E(Y)\) is so close to 0, \(E(Y) < 0.017\) for \(Y < 1\).

### 4 Bézier fitting

A Bézier fitting was born circa 50 years ago in the modelling of essentially curvilinear shapes. Since then it is widely used to produce the equations of curves (and, more generally, surfaces) by setting so-called control points.

This applicability takes advantage from de Casteljeau algorithm – the recursive recipe to evaluate the points sitting on so-called Bézier curves. This numerically stable algorithm was found by Pierre de Casteljeau in 1957, then 27 years old and working for Citroën (and the factory kept this algorithm in secret till 1975).

Discussed curves and surfaces are named after Pierre Étienne Bézier (1910–99), the mathematician for 42 years working for Renault.

From the mathematical point of view the Bézier fitting is simply the application of Bernstein polynomials to get parametrical equations describing a given curve (or, in 3D case, a given surface). Bernstein polynomials were defined by Sergiei Natanovich Bernstein in 1912 in his constructive proof of the Weierstrass approximation theorem. In 1885 Karl Weierstrass stated that every real-valued continuous function defined in any finite segment of the real line can be uniformly approximated as good as wished by a polynomial.

Let’s present two approximations. The first one, see Fig.3, gives the curve described by the parametrical equations

\[
x = 46.5t^6 - 42t^5 - 66t^4 + 118t^3 - 64.5t^2 + 18t - 1,
\]

\[
y = 0.7t^6 - 0.3t^5 - 2.4t^4 + 2.4t^3 - 0.15t^2,
\]

for \(t\) running from 0 to 1 it approximates the arc of the curve \(E = E(Y)\) run from \(Y = 10^{-1.8}\) to \(10^0\), resp. In Fig.4 you see two Bézier curves. They smoothly meet at the point \((Y, E) = (10^{1.8}, 1)\). They are generated by control points listed in the legenda to Fig.4. The left one is covered by the equations

\[
x = 3.16t^3 - 6.36t^2 + 6t - 1,
\]

\[
y = 0.145t^3 - 0.045t^2,
\]

the right Bézier curve has the parametrical equation

\[
x = 4.8t^3 - 1.5t^2 + 3.9t + 1.8,
\]

\[
y = 0.18t^3 - 0.51t^2 + 0.48t + 0.1.
\]
In both cases \( t \) runs the standard interval, \( 0 \leq t \leq 1 \), and they start at points \((Y, E) = (10^{-1}, 0)\) and \((10^{1.8}, 0.1)\), respectively.

Both these fits are elegant and concise, but because of their parametrical form they are hard to be applied in an engineer’s practice, where there is a need to have the explicit formula for \( E(Y) \). Obviously, it may happen that the dependencies \( x = x(t) \) have simple inversions, but here it does not even for third degree polynomials.

5 Spline fit

The notion ‘spline’ was introduced by I.J. Schoenberg, in the paper *Contributions to the problem of approximation of equidistant data by analytic functions. Part A – on the problem of smoothing or graduation. A first class of analytic approximation formulae* (1946), but its author asserts that the concept of the spline was known to P.S. Laplace. There are I.J. Schoenberg and L. Schumaker who developed the theory of splines and showed the applications of these ‘piecewise polynomials’ in various fields, in particular in approximation. The triumphal entrance of splines to the practice took place in 1973 when J.G. Herriot and C.H. Reinsch published *Algorithm 472: procedures for natural spline interpolation* – the recipe to produce a collocation spline of the third degree, i.e., composed of arcs of polynomials of the third degree and joint at given points called collocation nodes (see, e.g. [11]).

One can apply Herriot-Reinsch algorithm (HeRa) to get the spline collocated at points having abscissa \( x = 0, 1.4, 2.2, 4, \) and 7, i.e. \( Y = 1, 25.2, 158, 10^9\), and \( 10^7\), resp.; ordinate of these points are, clearly, equal to \( E(Y) \). Putting \( S(x) = 0 \) for \( x \leq 0 \) and \( S(x) = 0.25 \) for \( x \geq 7 \) and carefully rounding the coefficients produced by HeRa we obtain the following spline

\[
S(x) = 0 \text{ for } x \leq 0, \text{ i.e., if } Y \leq 1, \\
S(x) = 0.0268x^3 - 0.0126x^2 + 0.00273x \\
\text{for } x \in (0, 1.4], \\
\text{i.e., if } 1 < Y \leq 25.2, \\
S(x) = -0.0101x^2 + 0.153x - 0.142 \\
\text{for } x \in (1.4, 2.2], \\
\text{i.e., if } 25.2 < Y \leq 158, \\
S(x) = 0.00491x^3 - 0.0692x^2 + 0.333x - 0.304 \\
\text{for } x \in (2.2, 4], \\
\text{i.e., if } 158 < Y \leq 10^4, \\
S(x) = 0.00085x^3 - 0.0162x^2 + 0.104x + 0.0241 \\
\text{for } x \in (4, 7), \\
\text{i.e., if } 10^4 < Y < 10^7, \\
S(x) = 0.25 \text{ for } 7 \leq x, \text{ i.e., if } Y > 10^7.
\]

As before, the correspondence between ends of intervals for \( x \) and \( Y \) are approximate. At the points \( x = 0, 1.4, 2.2, 4, \) and 7, where neighbouring intervals meet, we can take use the formula corresponding to any of these intervals, the difference, in absolute value magnified by \( 10^4\), does not exceed \( 0, 4, 3, 3 \) and 3, respectively, so in engineering practice the spline \( S \) is a continuous function. The quality of the approximation of \( E = E(x) \) by \( E = S(x) \) is also very high (see Fig.5): at every point \( t \) of 61 points regularly distributed in the interval \((-2, 10)\) the error \( |E(t) - g(t)| < 10^{-4}\); also the common eye inspection does not distinguish the graphs \( E = E(x) \) and \( E = S(t) \).

6 Conclusions

It is hard to expect an engineer, as well as a designer or a researcher investigating the Bingham flow along a pipeline, to use the formula (4) in the aim to calculate the correction term \( \varepsilon = \varepsilon(y) \) changing the solution of the simplified Buckingham-Reiner equation (2) to the exact solution of the equation (1). It is inconvenient, and usually error-charged, to read the values \( \varepsilon \) from the graph. Therefore it is worth to have a simpler formula for \( \varepsilon \). Knowing that some approaches (such as, e.g., the standard Lagrange interpolation) are useless, in the paper we discuss approximations by logistic functions, Bézier curves and splines (including their combinations). Having them determined on the computer, we carefully round many-digit coefficients to have less decimal digits (we take into account the practical use of these formulas and that’s why we do the rounding). Among produced expressions we recommend to use the rounded spline approximation (20) produced by HeRa and having coefficients of polynomials appropriately rounded.
Our results is engineer-friendly and goes well with other recent results showing that the spline approximation is usually found to be the best in other problem considered in rheology (see, e.g. [1], [9], [13] and [15], where there is considered a sigmoid behavior similar to the one discussed here). Let us mention that the approximation of sigmoidal-shape behaviour is still in focus in various engineering scientific investigations, see, e.g. [12],[15],[17].

References: