Stochastic parametric resonance of a fractional oscillator

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Abstract: Underdamped motion of a fractional oscillator with fluctuating eigenfrequency subjected to an internal additive fractional noise is investigated. A friction kernel of the viscoelastic type, with memory, is assumed as a power-law function of time. The exact formula for the variance of the output signal in the long-time limit is derived. It is established that of intermediate values of the memory exponent, the energetic stability of the oscillator is enhanced in comparison with the cases of strong and low memory. Dependence of such an anomalous property of the oscillator on other system parameters is also discussed.

Key–Words: Stochastic oscillator, fractional oscillator, stochastic resonance, fractional noise, white noise, energetic instability.

1 Introduction

Recently, noise-induced phenomena in complex systems have been the topic of a number of physical investigations in different fields ranging from ecosystems [1, 2] to intracellular protein transport in biology [3, 4], or to methods of particle separation in nanotechnology [5, 6]. One of the objects of special attention in this context is the noise-driven harmonic oscillator [7]-[10]. The harmonic oscillator is the simplest toy model for different phenomena in nature and as such it is a typical theoretician’s paradigm for various fundamental conceptions [11]. It has been shown that the influence of noise on the oscillator frequency may lead to energetic instability, which manifests itself in an unlimited increase of second-order moments of the output with time, while the mean value of the oscillator displacement remains finite [8, 12]. This phenomenon is a stochastic counterpart of classical parametric resonance [8, 13]. A popular generalization of the harmonic oscillator consists in the replacement of the usual friction term in the dynamical equation for a harmonic oscillator by a generalized friction term with a power law type memory [14]-[16]. The main advantage of this equation is that it provides a physically transparent and mathematically tractable description of the stochastic dynamics in a system with slow relaxation processes and with anomalously slow diffusion (subdiffusion). Examples of such systems are supercooled liquids, glasses, colloidal suspensions, dense polymer solutions [17, 18], viscoelastic media [19], and amorphous semiconductors [20].

Although the behavior of both above-mentioned generalizations of the harmonic oscillator, i.e., a harmonic oscillator with fluctuating frequency and a fractional oscillator, are investigated in detail, it seems that analysis of the potential consequences of interplay between eigenfrequency fluctuations and memory effects is still missing in literature. This is quite surprising because the importance of fluctuations and viscoelasticity for biological systems, e.g., in living cells, has been well recognized [21].

Thus motivated, we consider a fractional oscillator with a power-law memory kernel. The influence of the fluctuating environment is modeled by a multiplicative white noise (fluctuating eigenfrequency) and by an additive fractional noise with a zero mean.

The main contribution of this paper is as follows. We provide exact formulas for the analytic treatment of the dependence of the variance of the output signal in the long-time limit, $t \to \infty$, on system parameters. On the basis of those exact expressions we will show that the energetic stability of the oscillator is significantly enhanced at intermediate values of the memory exponent. To our knowledge, the resonance-like dependence of the critical noise intensity, above which the oscillator is energetically unstable, on the memory exponent is a new effect that has never been discussed before.

The structure of the paper is as follows. In Section 2 we present, in a general form, the model investigated. Using Kubo’s second fluctuation-dissipation theorem, some general formulas for the variance of the oscillator displacement are derived in Section 3. In Section 4 we introduce a friction kernel with a power-
law type memory and exact formulas are found for the analysis of the long-time behavior of the variance. The phenomenon of memory–enhanced energetic stability is exposed and discussed in Section 5. Section 6 contains some brief concluding remarks.

2 Model

As a model for an oscillatory system strongly coupled with a noisy environment, we consider an oscillator with a fluctuating eigenfrequency and with a memory friction kernel

$$\dot{X}(t) + \int_0^t \gamma(t-t') \dot{X}(t') dt' + [\omega^2 + Z(t)]X(t) = \xi(t),$$

(1)

where $\dot{X} \equiv dX/dt$, $X(t)$ is the oscillator displacement, and fluctuations of the eigenfrequency $\omega$ are expressed as a Gaussian white noise $Z(t)$ with a zero mean and a delta-correlated correlation function:

$$\langle Z(t)Z(t') \rangle = 2D\delta(t-t'),$$

(2)

where $D$ is the noise intensity. The random driving force $\xi(t)$ characterizes the influence of fluctuations of a non-Ohmic thermal bath (environment) on the oscillator. In the following, we will treat the random force $\xi(t)$ as a zero-centered internal noise, i.e., we assume that the fluctuations and dissipation stem from the same source and the system will finally reach an equilibrium state. In this case the frictional kernel $\gamma(t)$ is related to the correlation function of the noise via Kubo’s second fluctuation–dissipation theorem [22]

$$\langle \xi(t)\xi(t') \rangle = k_B T \gamma(|t-t'|),$$

(3)

where $T$ is the absolute temperature, and $k_B$ is the Boltzmann constant. Here we emphasize that the noise $Z(t)$ in Eq. (1) is regarded as an external noise (e.g., caused by another independent thermal bath). In this case, the fluctuations and dissipation come from different sources, and the frictional kernel and the correlation function of the noise are independent.

3 General analysis of variance

By applying the Laplace transformation to Eq. (1), one can easily obtain a formal expression for the displacement $X(t)$ and the velocity $\dot{X}(t)$ in the following forms:

$$X(t) = \langle X(t) \rangle + \int_0^t H(t-\tau) [\xi(\tau) - X(\tau)Z(\tau)] d\tau$$

(4)

and

$$\dot{X}(t) = \langle \dot{X}(t) \rangle + \int_0^t \dot{H}(t-\tau) [\xi(\tau) - X(\tau)Z(\tau)] d\tau,$$

(5)

where the averages (over realizations of the stochastic processes $\xi(t)$ and $Z(t)$) $\langle X(t) \rangle$ and $\langle \dot{X}(t) \rangle$ are given by

$$\langle X(t) \rangle = \dot{x}_0 H(t) + x_0 \left[1 - \omega^2 \int_0^t H(\tau) d\tau \right],$$

(6)

$$\langle \dot{X}(t) \rangle = \dot{x}_0 \dot{H}(t) - \omega^2 x_0 H(t),$$

(7)

with deterministic initial conditions such as $X(0) = x_0$ and $\dot{X}(0) = \dot{x}_0$. The kernel $H(t)$ with the initial condition $H(0) = 0$ is the Laplace inversion of

$$\dot{H}(s) = \frac{1}{s^2 + s\gamma(s) + \omega^2},$$

(8)

where

$$\gamma(s) = \int_0^\infty e^{-st} \gamma(t) dt.$$
Moreover, the first term in the right side of Eq. (13) can be simplified [24] to
\[
\int_0^t \int_0^t H(t_1)H(t_2)\gamma(|t_1 - t_2|)dt_1dt_2 =
\]

\[
2 \int_0^t H(t_1)dt_1 - H^2(t) - \omega^2 \int_0^t H(t_1)dt_1 \right)^2 . \quad (14)
\]

Let us now assume that at a long-time limit, \( t \to \infty \), the kernel \( H(t) \) tends to zero sufficiently rapidly and that \( X(t) \) tends to a stationary process. In this case \( \lim_{s \to 0} (s^\gamma(s)) = 0 \) and
\[
\int_0^\infty H(t)dt = \frac{1}{\omega^2} .
\]

Thus, from Eqs. (6) and (7) it follows that \( \langle X(\infty) \rangle = \langle \dot{X}(\infty) \rangle = 0 \) and the stationary variance \( \sigma^2 := \sigma^2(\infty) \) determined by Eqs. (13) and (14) can be given as
\[
\sigma^2 = \frac{k_B T D_{cr}}{\omega^2(D_{cr} - D)} , \quad (15)
\]

where the critical noise intensity \( D_{cr} \) reads
\[
D_{cr} = \left[ 2 \int_0^\infty H^2(t)dt \right]^{-1} . \quad (16)
\]

From Eq. (15) we can see that the stationary regime is possible only if \( D < D_{cr} \). As the noise intensity \( D \) tends to the critical value \( D_{cr} \), the variance \( \sigma^2 \) increases to infinity. This is the indication that for \( D > D_{cr} \) energetic instability appears, which manifests itself in an unlimited increase of second-order moments of the output of the oscillator with time, while the mean value of the oscillator displacement remains finite [8, 12]. This phenomenon is a stochastic counterpart of classical parametric resonance in the case of ordinary deterministic oscillators (without a memory kernel) with time-dependent frequencies [8, 13].

## 4 Fractional oscillator

To find more physically tractable results, we introduce a noise \( \xi(t) \), known as fractional Gaussian noise, in Eq. (1). The fractional Gaussian noise is closely related to the fractional Brownian motion process [15, 25]. The last process has two unique properties: self-similarity with the Hurst coefficient \( 0 < H < 1 \), and stationary increments [26]. In the case of the fractional Gaussian noise with \( 1/2 < H < 1 \)
\[
\langle \xi(t+\tau)\xi(t) \rangle = \frac{k_B T \gamma \tilde{\omega}^{1-\alpha}}{\Gamma(1-\alpha)|\tau|^\alpha} , \quad 0 < \alpha < 1 , \quad (17)
\]

where the memory exponent \( \alpha = 2 - 2H \), \( \Gamma(z) \) is the gamma function, and \( \tilde{\omega} \) denotes a reference frequency allowing for the friction constant \( \gamma \) to have the dimension of viscosity of any values of the memory exponent. Thus Eq. (1) behaves as a stochastically perturbed fractional oscillator
\[
\frac{d^2X}{dt^2} + \gamma \tilde{\omega}^{1-\alpha} \frac{d^\alpha X}{dt^\alpha} + [\omega^2 + Z(t)]X(t) = \xi(t) , \quad (18)
\]

where
\[
\frac{d^\alpha X}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{X}(t')}{(t-t')^\alpha} dt' . \quad (19)
\]

is the fractional Caputo derivative [27]. Note that at the same time, Eq. (18) corresponds also to the sub-Ohmic model of coupling to the thermal bath oscillators [28] or to a fracton thermal bath [29].

Now we consider the dependence of the critical noise intensity \( D_{cr} \) on the system parameters. Using the Eqs. (3), (17), and (9) we find from Eq. (8) that
\[
\hat{H}(s) = \frac{1}{s^2 + \gamma \tilde{\omega}^{1-\alpha}s^\alpha + \omega^2} . \quad (20)
\]

The inverse Laplace transform gives (see [30])
\[
H(t) = \frac{2}{\sqrt{u^2 + v^2}} e^{-\beta t} \sin(\Omega t + \phi)
+ \frac{\gamma \tilde{\omega}^{1-\alpha} \sin(\alpha \pi)}{\pi} \int_0^\infty r^{-\alpha} e^{-rt} dr \frac{dr}{B(r)} , \quad (21)
\]

where \( s_{1,2} = -\beta \pm i\Omega \ (\beta > 0, \ \Omega > 0) \) are the pair of conjugate complex zeros of the equation
\[
A(s) \equiv s^2 + \gamma \tilde{\omega}^{1-\alpha} s^\alpha + \omega^2 = 0 ; \quad (22)
\]

\( A(s) \) is defined by the principal branch of \( s^\alpha \). The quantities \( u, v, \) and \( \phi \) are determined by
\[
u = -2\beta + \frac{\alpha \gamma \tilde{\omega}^{1-\alpha} \cos[(1 - \alpha) \arctan(-\Omega/\beta)]}{(\beta^2 + \Omega^2)^{1/2}},
\]

\[
v = 2\Omega - \frac{\alpha \gamma \tilde{\omega}^{1-\alpha} \sin[(1 - \alpha) \arctan(-\Omega/\beta)]}{(\beta^2 + \Omega^2)^{1/2}},
\]

\[
\phi = \arctan \left( \frac{u}{v} \right) . \quad (23)
\]
and
\[ B(r) := \left[ r^2 + \gamma \tilde{\omega}^{1-\alpha} r^\alpha \cos(\pi \alpha) + \omega^2 \right]^2 + \gamma^2 \omega^{2(1-\alpha)} r^{2\alpha} \sin^2(\alpha \pi). \] (24)

Then, from Eq. (16),
\[ D_{cr}^{-1} = \frac{2}{\beta(\beta^2 + \Omega^2)[u^2 + v^2]^2} \times \{(\beta^2 + \Omega^2)(u^2 + v^2) - \beta[\beta(v^2 - u^2) - 2\Omega uv]\} + \frac{4 \gamma^2 \omega^{2(1-\alpha)} \sin^2(\alpha \pi)}{\pi} \int_0^\infty \frac{r^\alpha dr}{B(r)} \int_0^r \frac{s^\alpha ds}{(r + s)B(s)} + \frac{8 \gamma \tilde{\omega}^{1-\alpha} \sin(\alpha \pi)}{\pi(u^2 + v^2)} \int_0^\infty \frac{r^\alpha [v\Omega + u(r + \beta)]dr}{[\Omega^2 + (r + \beta)^2]B(r)}. \] (25)

It must be emphasized that the relaxation function \( H(t) \) as well as \( D_{cr}^{-1} \) can be represented via Mittag-Leffler-type special functions [27]. But as the numerical calculations in this case are very complicated and apart from possible representations via Mittag-Leffler functions we suggest a numerical treatment of Eq. (25).

5 Memory enhanced energetic stability

In this section we investigate, on the basis of Eq. (25), the dependence of the appearance of energetic instability on the memory exponent \( \alpha \) at various values of the friction coefficient \( \gamma \) and a reference frequency \( \tilde{\omega} \). Here we emphasize that for all figures we use a dimensionless formulation of the dynamics with a time scaling of the following form:
\[ t^* = \omega t, \quad \gamma^* = \frac{\gamma}{\omega}, \quad \tilde{\omega}^* = \frac{\tilde{\omega}}{\omega}, \quad D^* = \frac{D}{\omega^3}, \] (26)
i.e. \( \omega^* = 1 \).

It is well known that in the case of an ordinary oscillator (without memory, \( \alpha = 1 \)) with a fluctuating frequency the critical noise intensity \( D_{cr} = \gamma \omega^2 \). Thus, when the noise intensity exceeds the value \( \gamma \omega^2 \), the oscillator is energetically unstable [8]. In Fig. 1, several graphs depict the behavior of \( D_{cr}/\gamma \) versus the memory exponent \( \alpha \) for different representative values of the parameters \( \gamma \) and \( \tilde{\omega} \). These graphs show a typical resonance-like behavior of \( D_{cr}(\alpha) \). As a rule, the maximal value of \( D_{cr}/\gamma \) increases as the values of the friction coefficient \( \gamma \) or the reference frequency \( \tilde{\omega} \) increases, while the positions of the maxima are monotonically shifted to a lower \( \alpha \) with a rise in \( \gamma \).

Figure 1: Critical noise intensity \( D_{cr} \) versus the memory exponent \( \alpha \) at several values of the parameters \( \gamma \) and \( \tilde{\omega} \). All quantities are dimensionless with a time scaling \( \omega = 1 \). Solid line, \( \gamma = 1.62 \); dashed line, \( \gamma = 5 \); dotted line, \( \gamma = 0.8 \). Panel (a): \( \tilde{\omega} = 2/3 \); panel (b): \( \tilde{\omega} = 1 \); panel (c): \( \tilde{\omega} = 2 \).

Moreover, for some values of the parameters \( \gamma \) and \( \tilde{\omega} \) a multiresonance with two maxima appears [see the solid lines in Fig. 1]. Thus, at intermediate values of the memory exponent \( \alpha \) the energetic stability of the fractional oscillator is significantly enhanced in comparison with an ordinary oscillator (without memory, \( \alpha = 1 \)). The effect is very pronounced at high values of the damping \( \gamma \). A physical explanation for the behavior of \( D_{cr}(\alpha) \) in the case of strong memory,
\( \alpha \to 0 \), is based on the cage effect [31]. For small \( \alpha \) the friction force induced by the medium is not just slowing down the particle but also causing the particle to develop a rattling motion. To see this consider Eq. (18) together with Eq. (19) in the limit \( \alpha \to 0 
abla \alpha
\)

\[
\dot{X} + [\gamma \dot{\omega} + \omega^2 + Z(t)]X = \xi(t) + \gamma \dot{\omega} X(0). \quad (27)
\]

Equation (27) describes a stochastically perturbed harmonic motion with the zero value effective friction coefficient, \( \gamma_{ef} = 0 \), and with the effective eigenfrequency \( \omega_{ef} = \sqrt{\gamma \dot{\omega} + \omega^2} \). Evidently, the corresponding \( D_{cr} = \gamma_{ef} \omega_{ef}^2 \) tends to zero. In this sense the medium is binding the particle, preventing dissipation but forcing oscillations.

6 Conclusions

In the present work, we have analysed the phenomenon of stochastic parametric resonance within the context of a noisy, fractional oscillator with a fluctuating eigenfrequency driven by an additive fractional noise.

According to Kubo’s second fluctuation-dissipation theorem, a viscoelastic friction kernel with memory is assumed as a power-law function of time. The Laplace transformation technique allows us to find exact formulas for the variance of time. The Laplace transformation technique with memory is assumed as a power-law function of time.

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