Parameters Design Method of Delta Modulations with Non-uniform Sampling Periods

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Abstract: - A suggestion for design of optimum parameters of delta modulators with adapting the sampling frequency and fixed quantization step size have been present. The analysis have been based on the Abate analytical considerations and application of the special function: Lambert W.

Key-Words: Adaptive delta modulation, non-uniform sampling, sampling frequency adaptation, quantization step size, LambertW function.

1 Introduction
In the seventies (the 20th Century) the first system solutions of 1-bit delta converters with the fixed step size and the sampling frequency adaptation NS-DM (Non-uniform Sampling Delta Modulation) and next, the converters with adapting of both: step size as well as sampling frequency, named ANS-DM (Adaptive Non-uniform Sampling Delta Modulation) were suggested [1, 2]. In this solutions the number of the sampling frequency values was limited to two and the number of the step size from two to several dozen. Thus whole the dynamic range (DR) was obtained practically due to the adaptation of the step size.

One of the basic problem of the correct input signal approximation is an appropriate choice of the function that modify the adapted parameters. At the required great dynamic ranges of the input signal, external converter parameters (SNRmax, DR, BRavg) are similar, while applying the exponential and linear changes [3]. On the other hand the number of steps when applying the exponential function of the step size changes is distinctly smaller. Then, this function is applied more frequently in practice.

Zhu [1, 2, 4] presents the most detailed elaborations concerning the algorithm of non-uniform sampling with fixed step size.

2 Principle of the NS-DM technique
The NS-DM schemes have been proposed and studied in [1, 3]. This modulation is a method adapting the sampling frequency to the input signal variation. The block diagram of its idea is presented in Fig.1.

For the input signal $x(t)$ the staircase in the NS-DM modulator can be expressed as in (1):

$$s(t_i) = s(t_0) + \sum_{n=1}^{i} \Delta d_n$$

where: $d_i = \text{sgn}[x(t_i) - s(t_i)]$

and the output code stream is as in the equation (2):

$$b_i = \begin{cases} 
1 & \text{for } d_i = 1 \\
0 & \text{for } d_i = -1 
\end{cases}$$

Where: $t_i$ is the sampling instant, and $s(t_i)$ is the coded signal $x(t)$, at $t_i$ with the step size $\Delta$. The sampling intervals $\tau_i = t_{i+1} - t_i$ vary according to the characteristics of $x(t)$ and the next sampling time $t_{i+1}$ can be expressed as:

$$t_{i+1} = \tau_i + t_i$$
And sampling interval $\tau_s$

$$\tau_s = \begin{cases} P \cdot \tau_{s,1} & \text{for } b_{i-2} = b_{i-1} = b_i \\ Q \cdot \tau_{s,1} & \text{for } b_{i-2} = b_i \neq b_{i-1} \\ \tau_0 & \text{for other } b_{i-2}, b_{i-1}, b_i \end{cases}$$  \hspace{1cm} (4)$$

where: $P, Q$ are constant factors of sampling interval modification: $P \leq 1 \leq Q$.

Two other parameters establish a varying sampling intervals border. The $\tau_{\text{max}}$ is the upper bound of the sampling period change and $\tau_{\text{min}}$ is the lower bound.

The $\tau_0$ is called also start sampling interval and its value decides about the average output bit rate. The algorithm (4) can be presented as a frequency modification function and then parameters assume the name $f_{s,\text{min}}, f_{s,\text{max}}$ and $f_{s,\text{start}}$ [3].

Formula (4) represents the 3-bit sampling interval change adaptation algorithm. It is also described by the Modify Interval Function (MIF) table (<1 means increase frequency, >1 decrease frequency, 1 denotes a comeback to start frequency).

<table>
<thead>
<tr>
<th>$b_{i-2}$</th>
<th>$b_{i-1}$</th>
<th>$b_i$</th>
<th>MIF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>&lt;1</td>
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<tr>
<td>0</td>
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<td>0</td>
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<td>1</td>
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<td>&lt;1</td>
</tr>
</tbody>
</table>

One can see that the NS-DM output binary stream carries the information of not only the changing trend of the source signal but also the sampling timing of the modulator. So that in the demodulation process with the same algorithm the irregular staircase signal can be recovered [3].

### 3 Model of the input process and dependencies describing the quantization noise of delta modulators

The investigations [1, 3, 5] helped to determine the based dependencies describing the quantization noise of delta modulators when assuming the input speech signal as a noise having an integrated power spectrum (white spectrum band-limited to $\omega_m$ and Gaussian probability density function).

Quantization distortions of delta modulations are often estimated on the basis of the so-called Abate condition [3]. Owing to the introduction of several simplifying assumptions in the analysis of the delta modulation noise, Abate establish that the 1-bit delta coding of the stationary stochastic input process ensures the minimum of quantization noise (granular noise and overload noise) if the following condition is satisfied:

$$s = \frac{kB}{\chi \sqrt{S}} = \ln(2B)$$  \hspace{1cm} (5)

where:

- $s$ – slope loading factor,
- $B = f_i/2f_o$ – oversampling ratio,
- $BR_{\text{avg}}$ – average bit rate of NS-DM and ANS-DM delta modulation,
- $f_i$ – sampling frequency,
- $f_o = 3.4\text{kHz}$ – maximum frequency for telephony speech,
- $k$ – quantization step size,
- $\chi$ – constant determining input process,
- $S$ – mean signal power.

Analyzing the dependence (5) it ought to be stated that the minimum value of the quantization noise power of delta modulation depends on the sampling frequency and, moreover, on the type and value ($S$) of the input signal. In order to obtain this minimum, the step size $k$ (at a given input level and the sampling frequency $f_i$) ought to be chosen in such a way as to satisfy the condition (5).

From the condition (5) there comes a simple conclusion concerning all 1-bit adaptive delta modulations (ADM). To obtain the constant maximum $SNR$ value each change of the input level must be accompanied by an adequate change of step size $k$ and/or sampling frequency $f_i$. The LDM modulation ensures minimization of the quantization noise only for one input level since $k=\text{const}$ and $f_i=\text{const}$.

The maximum $SNR$ value in the LDM modulation for the white spectrum band-limited input process according to Abate [3] and Taub [6] amounts to:

$$SNR_{\text{max}} = S \left[ \frac{B^3}{0.194(\ln B)^2 + 0.4\ln B + 0.227} \right]$$  \hspace{1cm} (6)

It ought to be emphasized that the $SNR_{\text{max}}$ value does not depend on the step size. This step decides only at which input level this minimum will be reached (Fig.2).

### 4 Optimal step size value in NS-DM modulation

Determination of the NS-DM optimum step size helps to obtain the appropriate output parameters of coding technique. In [4, 5] a method of determination of
the optimum step size for the speech signal\(^1\) has been suggested at the criterion of quality consisting in maintenance \(\text{SNR} = \text{const}\). As it was shown many times before [7, 8], the sampling frequency \(f_s\) has a decisive influence on the \(\text{SNR}_{\text{max}}\) value. In the case of the NS-DM modulation it is necessary to provide a sufficient conversion quality (\(\text{SNR}_{\text{max}}\)) for the lowest input \(S_i\) by means of accepting the sufficient \(B_{\text{min}}\) value.

Step size \(k_{\text{opt}}\) that ensures obtaining minimum of quantization noise at the lowest input level \((S_i)\) can be calculated on the basis the dependence (5). For the assumed speech signal model the constant \(\chi\) equals 1,08 approximately. It means that:

\[
k_{\text{opt}} \equiv 1,08 \frac{\ln 2B_{\text{min}}}{B_{\text{min}}} \sqrt{\frac{S_1}{S_2}} \equiv 1,08 \frac{\ln 2B_{\text{max}}}{B_{\text{max}}} \sqrt{S_2} \quad (7)
\]

\(B_{\text{min}} = f_s_{\text{min}}/2f_o\) – minimal oversampling ratio,
\(B_{\text{max}} = f_s_{\text{max}}/2f_o\) – maximal oversampling ratio,
\(S_2, (S_i)\) – maximal (minimal) input signal level,

\(\text{DR} = S_2/S_1\) – dynamic range.

Using one of the expressions on the right side of the formula (7) depends on the fact which of remaining NS-DM converter parameters \((B_{\text{max}}, \text{DR}, \sqrt{S_2}, \sqrt{S_1})\) are given. Fig.2 illustrates an influence of the step size \(k\) value on the position of the lower and upper edge of the input signal dynamic range \(\text{DR}\) in which due to the sampling frequency adaptation, the NS-DM converter maintain the constant value of the \(\text{SNR}\) value. The step size value does not influence the \(\text{SNR}_{\text{max}}\) value. To determine step size value knowledge of \(\text{DR}\) value is it not enough, the maximum or minimum input signal level \((\sqrt{S_2}, \sqrt{S_1})\) is necessary. Owing to it the range of the input signal levels in which the \(\text{SNR}\) remains constant is precisely determined.

3.1 Boundary values of the step size

Positive, real value \(B\) satisfying the Nyquist condition that is the solution of the transcendental equation (5) at given \(\chi = 1,08\) and \(k = \text{const}\) is found in the simplest way by using the special function \(\text{LambertW}\) [9]. \(\text{LambertW}\) function has real positive solutions for \(x \in [0, 1/e]\) while \(x = \frac{k}{1,08 \sqrt{S}}\). After transformations, the following inequality is obtained:

\[
0 < k \leq 0,8 \sqrt{S} \quad (8)
\]

Inequality (8) has an important practical meaning pointing out to the designer the greatest value of the step size.

3.2. Design of NSDM companding parameters

The analysis carried out in [4, 5, 10] comprises a derivation of all indispensable dependences describing the NS-DM parameters. It helped to establish the correct sequence of individual parameters calculations.

\[\text{Input data:}\]
\[
\begin{align*}
S_1 & \quad \text{SNR} \\
S_2 & \quad \text{Calculate } f_{s_{\text{min}}} \\
S_2 & \quad \text{Calculate } f_{s_{\text{max}}} \\
S_1 & \quad \text{Calculate } k
\end{align*}
\]

\[\text{Output parameters}\]
\[
\begin{align*}
f_{s_{\text{max}}} & \quad f_{s_{\text{min}}} \quad k
\end{align*}
\]

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\(^1\) Input signal undergoing conversion, the so-called modeled speech signal is realized as a random process with the white spectrum band-limited to \(o_m\) and integrated by the RC filter that satisfies the dependence \(\frac{\alpha_m}{\alpha_o} = 0,23\).
Individual output parameters are determined on the basis of their definition and from equations (5÷7, 9, 10). Boundary frequencies of sampling ($f_{s_{min}}$ and $f_{s_{max}}$) are most easily determined while using LambertW function [11]:

$$B_{max} = \frac{-\ln(2B_{min})}{2B_{min} \sqrt{DR}}$$

$$B_{min} = \frac{-\sqrt{DR \ln(2B_{max})}}{2B_{max}}$$

In Figure 3 two simple examples of NS-DM companding parameters determination depending on the input data have been presented.

4 Conclusion

The presented principles of the NS-DM parameters design also concern ANS-DM modulation. It helps to determine appropriate parameters indispensable for their correct functioning in analytical way.

The form of the Abate dependence describing the minimum condition of quantization noise makes the special LambertW function extremely suitable for the determination of the maximum and minimum sampling frequencies ($f_{s_{min}}$ and $f_{s_{max}}$).

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References:


