# An Anisochronic Model of a Laboratory Heating System

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*Abstract:* - The paper is focused on an anisochronic modelling principle utilized on a laboratory heating system. Anisochronic models are characterized by that they exhibit state (internal) delays. The studied laboratory plant is designed as a thermal heating circuit system and it represents a small scale model with dynamic properties similar to those of real heating system (e.g. in cars). The obtained mathematical description will be used in the future research for the verification of algebraic control algorithms designed by the authors in their recent works.

*Key-Words:* - Anisochronic models, time-delay systems, modelling, heating systems.

## 1 Introduction

Heating systems still represent a favourite research area as it reveals from recent studies. To name a few, in [1] a method of integral equations for some thermal problems of engineering is proposed (e.g. for radiative heat transfer, heat conduction etc.) which leads to Volterra-Fredholm integrals. Pipelines and pipe connections are modelled in [2]. A model of heating system in a room, which is similar to that studied in this paper, is presented in [3]. In [4] a model incorporating internal delays (even of neutral type) for central heating system is presented. However, many of there approaches are a rather complicated and yield distributed parameter models.

The presented contribution deals with anisochronic modelling philosophy introduced already in [5] and subsequently developed for heating systems e.g. in [6-7]. Anisochronic models are characterized by the occurrence of state (internal) delays in a system model. Nevertheless, there are also many industrial processes that include delays in internal feedback loops, e.g. in the model of mass flows in sugar factory [8] or in metallurgic processes [9], to name a few.

The laboratory heating plant modelled in this paper was assembled at the Faculty of Applied Informatics of Tomas Bata University in Zlín in order to test control algorithms for systems with dead time. The original description of the apparatus and its electronic circuits can be found in [10]. The motivation for the modelling the plant was double. First, the dynamics of the plant exhibits unconventional step responses which cannot be explained by a standard analytic means. Second, the authors of this contribution intend to use the obtained anisochronic mathematical description of the plant with the view of the verification of algebraic control algorithms in the $R_{MS}$ ring designed for delayed systems, see e.g. [11-12]

## 2 Description of the Laboratory Heating Model

The plant to be mathematically modelled in this paper was built in our institute in order to verify several control algorithms for time delay systems. Originally, it was intended to control input delays only; however, as it is shown in this contribution, the plant contains internal delays as well, and thus it is suitable also for testing control approaches for anisochronic systems. The plant is based on the principle of transferring heat from a source through a piping system using a heat transferring media to a heat-consuming appliance. External appearance of the plant is shown in Fig.1.

Fig.1 – A photo of the laboratory heating model

A schematic sketch of the model is depicted in Fig.2

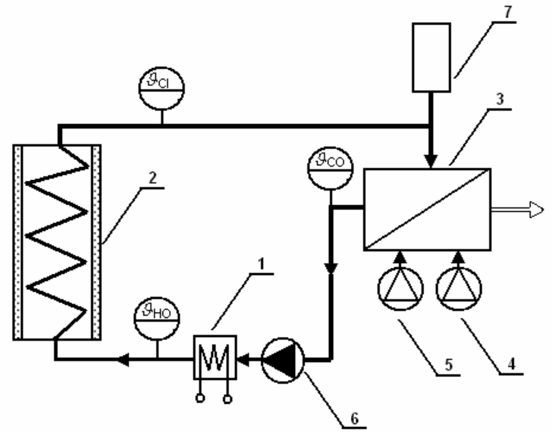

Fig.2 – A scheme of the laboratory heating model

The heat transferring fluid (e.g. water) is transported using a continuously controllable DC pump (6) into a flow heater (1) with max. power of 750 W. The temperature of a fluid at the heater output is measured by a platinum thermometer giving value of $\vartheta_{HO}$. Warmed liquid then goes through a 15 meters long insulated coiled pipeline (2) which causes the significant delay in the system. The air-water heat exchanger (3) with two cooling fans (4, 5) represents a heat-consuming appliance. The speed of the first fan can be continuously adjusted, whereas the second one is of on/off type. Input and output temperatures of the cooler are measured again by platinum thermometers as $\vartheta_{CI}$, resp. $\vartheta_{CO}$. The expansion tank (7) compensates for the expansion effect of the water. This small scale model can represents dynamics of real heating systems, e.g. a cooling system in cars, heating systems in buildings, etc. The laboratory model is connected to a standard PC via serial bus RS232 and a portable data acquisition unit. All tasks relating to the monitoring and control of the plant are served by software running in Matlab 6.5 environment.

# 3 The Anisochronic Model of the Plant

Obviously, an accurate mathematical model of the plant would be rather complicated due to the existence of components causing distributed delays in the system. However, the aim of this contribution is not to find an exact description of the model, but a sufficiently simple mathematical model which can be used for the verification of some control algorithms. Thus, in the following section the construction of a suitable anisochronic model is proposed. The methodology is based on comprehension of all significant delays and latencies in the model.

Let us introduce notation for process quantities first:

$c$ [J kg$^{-1}$ K$^{-1}$] – the specific heat capacity of water

$\dot{m}(t)$ [kg s$^{-1}$] – the mass flow rate of water

$M_H$ [kg] – the overall mass of water in the heater

$M_C$ [kg] – the overall mass of water in the cooler

$M_P$ [kg] – the overall mass of water in the pipeline

$\vartheta_{HO}(t)$ [°C] – output temperature of the heater

$\vartheta_{CI}(t)$ [°C] – input temperature of the cooler

$\vartheta_{CO}(t)$ [°C] – output temperature of the cooler

$\vartheta_{HI}(t)$ [°C] – input temperature of the heater

$\vartheta_A$ [°C] – ambient temperature

$P(t)$ [W] – the power of the heater

$\tau_H$ [s] – the delay of a water flow through the heater

$\tau_{HC}$ [s] – the delay of a water flow between the heater and the cooler

$\tau_C$ [s] – the delay of a water flow through the cooler

$\tau_{KC}$ [s] – the delay between a control signal to the cooling fan and the output temperature of the cooler

$\tau_{CH}$ [s] – the delay of a water flow between the cooler and the heater

$u_P(t)$ [V] – a voltage input to the pump

$u_C(t)$ [V] – a voltage input to the cooling fan

$K_H(t)$ [W K$^{-1}$] – the overall heat transmission coefficient of heater wastage energy

$K_C(t)$ [W K$^{-1}$] – the overall heat transmission coefficient of the cooler

$K_P$ [W K$^{-1}$] – the overall heat transmission coefficient of the long pipeline

$h_0$, $h_1$, $h_2$, $h_3$, $h_4$, $h_5$ – weighting coefficients for the estimation of the overall heat transmission coefficient of the heater

$c_0$ [W K$^{-1}$], $c_1$, $c_2$ – weighting coefficients for the estimation of the overall heat transmission coefficient of the cooler

$p_0$ [m$^3$ s$^{-1}$], $p_1$, $p_2$ – weighting coefficients for the estimation of the mass flow rate of water

## 3.1 A model of the heater

The energy balance equation is used for the description of the heater

$$cM_H \frac{\mathrm{d}\vartheta_{HO}(t)}{\mathrm{d}t} = P(t-0.5\tau_H) + c\dot{m}(t)[\vartheta_{HI}(t) - \vartheta_{HO}(t)] - K_H(t)\left[\frac{\vartheta_{HO}(t) + \vartheta_{HI}(t-\tau_H)}{2} - \vartheta_A\right] \quad (1)$$

where the arithmetical mean temperature difference is taken for heat losses. A heating body is assumed to perform heat energy in the middle of the heater. Input temperature, $\vartheta_{HI}(t)$, is estimated by "the nearest" measured one, $\vartheta_{CO}(t)$, as

$$\vartheta_{HI}(t) = \vartheta_{CO}(t - \tau_{CH}) \tag{2}$$

due to the fact that the fluid transport between the cooler output and the heater input is fast enough so that these two temperatures almost do not differ, except for a time delay. The overall heat transmission coefficient of the heater, $K_H(t)$, is numerically approximated by the relation

$$K_H(t) = \frac{h_0 P^2(t) + h_1 \dot{m}^2(t) + h_2 P(t)\dot{m}(t) + h_3}{h_4 P(t) + h_5 \dot{m}(t)} \tag{3}$$

see [13].

### 3.2 A model of the coiled insulated pipeline

A transportation delay in the piping has a decisive influence on the behaviour of the system. Consider the energy balance equation again where heat losses are supposed to be linear along the pipeline

$$\begin{aligned} cM_P \frac{\mathrm{d}\vartheta_{CI}(t)}{\mathrm{d}t} &= c\dot{m}(t)\left[\vartheta_{HO}(t - \tau_{HC}) - \vartheta_{CI}(t)\right] \\ &\quad - K_P\left[\frac{\vartheta_{CI}(t) + \vartheta_{HO}(t - \tau_{HC})}{2} - \vartheta_A\right] \end{aligned} \tag{4}$$

Notice that input and output temperatures are not considered in the same time since the thermal effect of the water inlet affects the outlet after some dead time.

The mass of the piping is neglected in the model due to the fact that the specific heat capacity of the material of the pipeline (copper) is much smaller than that of water, approximately ten times.

### 3.3 A model of the heat exchanger (cooler)

Time delays in the air-water exchanger are of a distributed nature, thus they have not an important role in system behaviour. On the other hand, the cooler significantly affects the temperature because of its high heat transmission coefficient supported by fans. The energy balance equation reads

$$\begin{aligned} cM_C \frac{\mathrm{d}\vartheta_{CO}(t)}{\mathrm{d}t} &= c\dot{m}(t)\left[\vartheta_{CI}(t - \tau_C) - \vartheta_{CO}(t)\right] \\ &\quad - K_C(t)\left[\frac{\vartheta_{CO}(t) + \vartheta_{CI}(t - \tau_C)}{2} - \vartheta_A\right] \end{aligned} \tag{5}$$

The dynamics of the air part of the cooler is much faster in comparison with the water one, thus this dynamics is neglected. The heat transmission coefficient, $K_C(t)$, is attempted to be approximated by a function

$$K_C(t) = c_2 u_C^2(t - \tau_{KC}) + c_1 u_C(t - \tau_{KC}) + c_0 \tag{6}$$

Changes in the fan speed affect the heat transmission coefficient, $K_C(t)$. Notice that there is a delay between the control input voltage to the continuously controllable cooling fan, $u_C(t)$, and a change of $K_C(t)$. There is no attempt to use models of all electrical and electronics equipments (e.g. the fan motor), and thus coefficients $c_0$, $c_1$, $c_2$ are determined experimentally.

### 3.4 A model of the pump

The influence of the voltage input to the pump, $u_P(t)$, upon the mass flow rate of water, $\dot{m}(t)$, can be described by a static characteristic

$$\dot{m}(t) = p_0\left[u_P(t) + p_1\right]^{p_2} \tag{7}$$

see [13]. The pump dynamics is omitted comparing to the whole process dynamics. Changes of process delays caused by the change of $\dot{m}(t)$ are neglected as well, in order to avoid a rather complicated mathematical description of the plant dynamics.

## 4 Model Linearization

Measured temperatures $\vartheta_{HO}(t)$, $\vartheta_{CI}(t)$, $\vartheta_{CO}(t)$ are taken as system outputs, whereas analog input voltages $u_P(t)$, $u_C(t)$ and the power $P(t)$ are considered as system inputs. Hence, multi-input multi-output (MIMO) model of a plant is obtained. Obviously, model (1)-(7) is nonlinear. To obtain linearized model, the first two terms of the Taylor series expansion at an operation point are used.

From (1)-(3) and (7)

$$\begin{aligned} \Delta\frac{\mathrm{d}\vartheta_{HO}(t)}{\mathrm{d}t} &= A_1 \Delta u_P(t) + \frac{1}{cM_H}\Delta P(t - 0.5\tau_H) \\ &\quad + A_2 \Delta P(t) + A_3 \Delta\vartheta_{HO}(t) + A_4 \Delta\vartheta_{CO}(t - \tau_H - \tau_{CH}) \end{aligned} \tag{8}$$

where

$$\begin{aligned} A_1 &= \frac{\mathrm{d}}{\mathrm{d}u_P(t)}\frac{\mathrm{d}\vartheta_{HO}(t)}{\mathrm{d}t}\bigg|_0 = \frac{p_0 p_2 (u_{P0} + p_1)^{p_2 - 1}}{M_H}\Bigg[\vartheta_{HI0} - \vartheta_{HO0} \\ &\quad + (0.5\vartheta_{HI0} + 0.5\vartheta_{HO0} - \vartheta_A)\Bigg(\frac{-h_1 h_5 p_0^2 (u_{P0} + p_1)^{2p_2}}{c\left[h_5 p_0 (u_{P0} + p_1)^{p_2} + h_4 P_0\right]^2} \\ &\quad + \frac{-2h_1 h_4 P_0 p_0 (u_{P0} + p_1)^{p_2} + (h_0 h_5 - h_2 h_4)P_0^2 - h_3 h_5}{c\left[h_5 p_0 (u_{P0} + p_1)^{p_2} + h_4 P_0\right]^2}\Bigg)\Bigg] \end{aligned}$$

$$A_2 = \frac{\mathrm{d}}{\mathrm{d}P(t)} \frac{\mathrm{d}\vartheta_{HO}(t)}{\mathrm{d}t}\bigg|_0 = \Bigg[ \vartheta_{HI0} - \vartheta_{HO0} + (0.5\vartheta_{HI0} + 0.5\vartheta_{HO0} - \vartheta_A) \frac{(h_1h_4 - h_2h_5)p_0^2(u_{P0}+p_1)^{2p_2} - 2h_0h_5P_0p_0(u_{P0}+p_1)^{p_2} - h_0h_4P_0^2 + h_3h_4}{cM_H\left[h_5p_0(u_{P0}+p_1)^{p_2} + h_4P_0\right]^2} \Bigg]$$

$$A_3 = \frac{\mathrm{d}}{\mathrm{d}\vartheta_{HO}(t)} \frac{\mathrm{d}\vartheta_{HO}(t)}{\mathrm{d}t}\bigg|_0 = -\frac{1}{M_H} \Bigg[ p_0(u_{P0}+p_1)^{p_2} + \frac{h_1p_0^2(u_{P0}+p_1)^{2p_2} + h_2P_0p_0(u_{P0}+p_1)^{p_2} + h_0P_0^2 + h_3}{2c\left[h_5p_0(u_{P0}+p_1)^{p_2} + h_4P_0\right]^2} \Bigg]$$

$$A_4 = \frac{\mathrm{d}}{\mathrm{d}\vartheta_{CO}(t-\tau_H-\tau_{CH})} \frac{\mathrm{d}\vartheta_{HO}(t)}{\mathrm{d}t}\bigg|_0 = \frac{1}{M_H} \Bigg[ p_0(u_{P0}+p_1)^{p_2} - \frac{h_1p_0^2(u_{P0}+p_1)^{2p_2} + h_2P_0p_0(u_{P0}+p_1)^{p_2} + h_0P_0^2 + h_3}{2c\left[h_5p_0(u_{P0}+p_1)^{p_2} + h_4P_0\right]^2} \Bigg]$$

Additional index $(\cdot)_0$ denotes the appropriate quantity value in the steady state (an operation point) and symbol $\Delta$ stands for deviation from an operation point. From (4) and (7) we have

$$\Delta \frac{\mathrm{d}\vartheta_{CI}(t)}{\mathrm{d}t} = A_5 \Delta u_P(t) + A_6 \Delta\vartheta_{HO}(t-\tau_{HC}) + A_7 \Delta\vartheta_{CI}(t) \qquad (9)$$

with

$$A_5 = \frac{\mathrm{d}}{\mathrm{d}u_P(t)} \frac{\mathrm{d}\vartheta_{CI}(t)}{\mathrm{d}t}\bigg|_0 = \frac{p_2p_0(u_{P0}+p_1)^{p_2-1}(\vartheta_{HO0} - \vartheta_{CI0})}{M_P}$$

$$A_6 = \frac{\mathrm{d}}{\mathrm{d}\vartheta_{HO}(t-\tau_H)} \frac{\mathrm{d}\vartheta_{CI}(t)}{\mathrm{d}t}\bigg|_0 = \frac{1}{cM_P}\left[cp_0(u_{P0}+p_1)^{p_2} - 0.5K_P\right]$$

$$A_7 = \frac{\mathrm{d}}{\mathrm{d}\vartheta_{CI}(t)} \frac{\mathrm{d}\vartheta_{CI}(t)}{\mathrm{d}t}\bigg|_0 = -\frac{1}{cM_P}\left[cp_0(u_{P0}+p_1)^{p_2} + 0.5K_P\right]$$

Linearization of (5)-(7) gives

$$\Delta \frac{\mathrm{d}\vartheta_{CO}(t)}{\mathrm{d}t} = A_8 \Delta u_P(t) + A_9 \Delta u_C(t-\tau_{KC}) + A_{10} \Delta\vartheta_{CO}(t) + A_{11} \Delta\vartheta_{CI}(t-\tau_C) \qquad (10)$$

$$A_8 = \frac{\mathrm{d}}{\mathrm{d}u_P(t)} \frac{\mathrm{d}\vartheta_{C0}(t)}{\mathrm{d}t}\bigg|_0 = \frac{p_2p_0(u_{P0}+p_1)^{p_2-1}(\vartheta_{CI0} - \vartheta_{CO0})}{M_C}$$

$$A_9 = \frac{\mathrm{d}}{\mathrm{d}u_C(t-\tau_{KC})} \frac{\mathrm{d}\vartheta_{C0}(t)}{\mathrm{d}t}\bigg|_0 = -\frac{(2c_2u_{C0}+c_1)p_2p_0(u_{P0}+p_1)^{p_2-1}(0.5\vartheta_{CI0} + 0.5\vartheta_{CO0} - \vartheta_A)}{cM_C}$$

$$A_{10} = \frac{\mathrm{d}}{\mathrm{d}\vartheta_{CO}(t)} \frac{\mathrm{d}\vartheta_{CO}(t)}{\mathrm{d}t}\bigg|_0 = -\frac{1}{cM_P}\left[2cp_0(u_{P0}+p_1)^{p_2} + c_2u_{C0}^2 + c_1u_{C0} + c_0\right]$$

$$A_{11} = \frac{\mathrm{d}}{\mathrm{d}\vartheta_{CO}(t-\tau_C)} \frac{\mathrm{d}\vartheta_{CO}(t)}{\mathrm{d}t}\bigg|_0 = \frac{1}{cM_P}\left[2cp_0(u_{P0}+p_1)^{p_2} - c_2u_{C0}^2 + c_1u_{C0} + c_0\right]$$

A linearized state space model in an operation point then reads

$$\begin{bmatrix} \frac{d}{dt}\Delta\vartheta_{HO}(t) \\ \frac{d}{dt}\Delta\vartheta_{CI}(t) \\ \frac{d}{dt}\Delta\vartheta_{CO}(t) \end{bmatrix} = \begin{bmatrix} A_3 & 0 & 0 \\ 0 & A_7 & 0 \\ 0 & 0 & A_{10} \end{bmatrix} \begin{bmatrix} \Delta\vartheta_{HO}(t) \\ \Delta\vartheta_{CI}(t) \\ \Delta\vartheta_{CO}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\vartheta_{HO}(t-\tau_{CH}-\tau_H) \\ \Delta\vartheta_{CI}(t-\tau_{CH}-\tau_H) \\ \Delta\vartheta_{CO}(t-\tau_{CH}-\tau_H) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ A_6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\vartheta_{HO}(t-\tau_H) \\ \Delta\vartheta_{CI}(t-\tau_H) \\ \Delta\vartheta_{CO}(t-\tau_H) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A_{11} & 0 \end{bmatrix} \begin{bmatrix} \Delta\vartheta_{HO}(t-\tau_C) \\ \Delta\vartheta_{CI}(t-\tau_C) \\ \Delta\vartheta_{CO}(t-\tau_C) \end{bmatrix}$$

$$+ \begin{bmatrix} A_1 & 0 & A_2 \\ A_5 & 0 & 0 \\ A_8 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta u_P(t) \\ \Delta u_C(t) \\ \Delta P(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{1}{cM_H} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta u_P(t-0.5\tau_H) \\ \Delta u_C(t-0.5\tau_H) \\ \Delta P(t-0.5\tau_H) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A_9 & 0 \end{bmatrix} \begin{bmatrix} \Delta u_P(t-\tau_{KC}) \\ \Delta u_C(t-\tau_{KC}) \\ \Delta P(t-\tau_{KC}) \end{bmatrix}$$

$$\begin{bmatrix} \Delta\vartheta_{HO}(t) \\ \Delta\vartheta_{CI}(t) \\ \Delta\vartheta_{CO}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\vartheta_{HO}(t) \\ \Delta\vartheta_{CI}(t) \\ \Delta\vartheta_{CO}(t) \end{bmatrix} \qquad (11)$$

It should be note that the whole system state is given not only by current values of state variables at time $t$, but also by a segment of last system history within the time range $\langle t-\tau, t\rangle$ where $\tau = \max\{\tau_C, \tau_{CH} + \tau_H\}$. Symbol $\Delta$ for the linearized model is omitted hereinafter. Assuming zero initial conditions (i.e. steady state in an operation point), the Laplace transform of (11) is given by

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$$
\begin{bmatrix}\Theta_{HO}(s)\\ \Theta_{CI}(s)\\ \Theta_{CO}(s)\end{bmatrix}s
=\begin{bmatrix}A_3 & 0 & A_4\exp(-(\tau_{CH}+\tau_H)s)\\ A_6\exp(-\tau_H s) & A_7 & 0\\ 0 & A_{11}\exp(-\tau_C s) & A_{10}\end{bmatrix}\begin{bmatrix}\Theta_{HO}(s)\\ \Theta_{CI}(s)\\ \Theta_{CO}(s)\end{bmatrix}
+\begin{bmatrix}A_1 & 0 & A_2\dfrac{\exp(-0.5\tau_H s)}{cM_H}\\ A_5 & 0 & 0\\ A_8 & A_9\exp(-\tau_{KC}s) & 0\end{bmatrix}\begin{bmatrix}U_P(s)\\ U_C(s)\\ P(s)\end{bmatrix}
\tag{12}
$$

where the capital letters stand for transformed variables denoted with corresponding lower case letters. The transfer matrix of the model thus reads

$$
\begin{bmatrix}\Theta_{HO}(s)\\ \Theta_{CI}(s)\\ \Theta_{CO}(s)\end{bmatrix}=\begin{bmatrix}G_{11}(s) & G_{12}(s) & G_{13}(s)\\ G_{21}(s) & G_{22}(s) & G_{23}(s)\\ G_{31}(s) & G_{32}(s) & G_{33}(s)\end{bmatrix}\begin{bmatrix}U_P(s)\\ U_C(s)\\ P(s)\end{bmatrix}
=\frac{1}{A(s)}\begin{bmatrix}B_{11}(s) & B_{12}(s) & B_{13}(s)\\ B_{21}(s) & B_{22}(s) & B_{23}(s)\\ B_{31}(s) & B_{32}(s) & B_{33}(s)\end{bmatrix}\begin{bmatrix}U_P(s)\\ U_C(s)\\ P(s)\end{bmatrix}
\tag{13}
$$

where

$$
\begin{aligned}
B_{11}(s)&=\beta_{11,2}s^2+\beta_{11,1}s+\beta_{11,1D}s\exp(-\tau_{11,1D}s)+\beta_{11,0}\\
&\quad+\beta_{11,0D1}\exp(-\tau_{11,0D1}s)+\beta_{11,0D2}\exp(-\tau_{11,0D2}s)\\
B_{12}(s)&=(\beta_{12,1}s+\beta_{12,0})\exp(-\tau_{12}s)\\
B_{13}(s)&=\beta_{13,2}s^2+\beta_{13,2D}s^2\exp(-\tau_{13}s)+\beta_{13,1}s\\
&\quad+\beta_{13,1D}s\exp(-\tau_{13}s)+\beta_{13,0}s+\beta_{13,0D}s\exp(-\tau_{13}s)\\
B_{21}(s)&=\beta_{21,2}s^2+\beta_{21,1}s+\beta_{21,1D}s\exp(-\tau_{21,1D}s)\\
&\quad+\beta_{21,0}+\beta_{21,0D1}\exp(-\tau_{21,0D1}s)+\beta_{21,0D2}\exp(-\tau_{21,0D2}s)\\
B_{22}(s)&=\beta_{22,0}\exp(-\tau_{22}s)\\
B_{23}(s)&=[\beta_{23,1}s+\beta_{23,1D}s\exp(-\tau_{23,1D}s)+\beta_{23,0}\\
&\quad+\beta_{23,0D}\exp(-\tau_{23,0D}s)]\exp(-\tau_{23}s)\\
B_{31}(s)&=\beta_{31,2}s^2+\beta_{31,1}s+\beta_{31,1D}s\exp(-\tau_{31,1D}s)+\beta_{31,0}\\
&\quad+\beta_{31,0D1}\exp(-\tau_{31,0D1}s)+\beta_{31,0D2}\exp(-\tau_{31,0D2}s)\\
B_{32}(s)&=(\beta_{32,2}s^2+\beta_{32,1D}s+\beta_{32,0})\exp(-\tau_{32}s)\\
B_{33}(s)&=[\beta_{33,0D}s\exp(-\tau_{33,0}s)+\beta_{33,0}]\exp(-\tau_{33}s)\\
A(s)&=s^3+\alpha_2s^2+\alpha_1s+\alpha_0+\alpha_{0D}\exp(-\tau_{0D}s)
\end{aligned}
$$

with

$\beta_{11,2}=A_1$, $\beta_{11,1}=-A_1(A_7+A_{10})$, $\beta_{11,1D}=A_4A_8$,
$\beta_{11,0}=A_1A_7A_{10}$, $\beta_{11,0D1}=-A_4A_7A_8$, $\beta_{11,0D2}=A_4A_5A_{11}$,
$\tau_{11,1D}=\tau_{11,0D1}=\tau_{CH}+\tau_H$, $\tau_{11,0D2}=\tau_C+\tau_{CH}+\tau_H$
$\beta_{12,1}=A_4A_9$, $\beta_{12,0}=-A_4A_7A_9$, $\tau_{12}=\tau_{KC}+\tau_{CH}+\tau_H$
$\beta_{13,2}=A_2$, $\beta_{13,2D}=\dfrac{1}{cM_H}$, $\beta_{13,1}=-A_2(A_7+A_{10})$,
$\beta_{13,1D}=-\dfrac{1}{cM_H}(A_7+A_{10})$, $\beta_{13,0}=A_2A_7A_{10}$,
$\beta_{13,0D}=\dfrac{1}{cM_H}A_7A_{10}$, $\tau_{13}=0.5\tau_H$
$\beta_{21,2}=A_5$, $\beta_{21,1}=-A_5(A_3+A_{10})$, $\beta_{21,1D}=A_1A_6$,
$\beta_{21,0}=A_3A_5A_{10}$, $\beta_{21,0D1}=-A_1A_6A_{10}$, $\beta_{21,0D2}=A_4A_6A_8$,
$\tau_{21,1D}=\tau_{21,0D1}=\tau_{HC}$, $\tau_{21,0D2}=\tau_{CH}+\tau_H+\tau_{HC}$
$\beta_{22,0}=A_4A_6A_9$, $\tau_{22}=\tau_{KC}+\tau_{CH}+\tau_H+\tau_{HC}$
$\beta_{23,1}=A_2A_6$, $\beta_{23,1D}=A_6\dfrac{1}{cM_H}$, $\beta_{23,0}=-A_2A_6A_{10}$,
$\beta_{23,0D}=-A_6A_{10}\dfrac{1}{cM_H}$, $\tau_{23}=\tau_{HC}$, $\tau_{23,1D}=\tau_{23,0D}=0.5\tau_H$
$\beta_{31,2}=A_8$, $\beta_{31,1}=-A_8(A_3+A_7)$, $\beta_{31,1D}=A_5A_{11}$,
$\beta_{31,0}=A_3A_7A_8$, $\beta_{31,0D1}=-A_3A_5A_{11}$, $\beta_{31,0D2}=A_1A_6A_{11}$,
$\tau_{21,1D}=\tau_{31,0D1}=\tau_C$, $\tau_{31,0D2}=\tau_{HC}+\tau_C$
$\beta_{32,2}=A_9$, $\beta_{32,1}=-A_9(A_3+A_7)$, $\beta_{32,0}=A_3A_7A_9$,
$\tau_{32}=\tau_{KC}$
$\beta_{33,0D}=A_6A_{11}\dfrac{1}{cM_H}$, $\beta_{33,0}=A_2A_6A_{11}$,
$\tau_{33,0}=0.5\tau_H$, $\tau_{33}=\tau_{HC}+\tau_C$
$\alpha_2=-(A_3+A_7+A_{10})$, $\alpha_1=A_3A_7+A_3A_{10}+A_7A_{10}$,
$\alpha_0=-A_3A_7A_{10}$, $\alpha_{0D}=-A_4A_6A_{11}$,
$\tau_{0D}=\tau_H+\tau_{HC}+\tau_C+\tau_{CH}$

## 5 Parameters Identification

Prior to solving the task of enumeration of model parameters, let us display how unconventional the step response of the system is. Consider the step change of $P(t)$ resulting in changes of system output temperatures, as it is pictured in Fig.3. An interesting feature of the step response is the existence of "stairs" ("quasi" steady states) in the plot.

The existence of these multiple "quasi" steady states can be explained as follows: Temperature of water at the heater output, $\vartheta_{HO}(t)$, increases until the energy inlet and outlet of the heater equal. In the meanwhile, the "hot" water flow goes through the long pipe to the cooler, and, after some dead-time, $\tau_{HC}$, it affects input, $\vartheta_{CI}(t)$, and output, $\vartheta_{CO}(t)$, temperatures of the cooler. At this time, the heater input temperature remains constant, because the water flow has not gone a round yet, and $\vartheta_{CO}(t)$ becomes constant. Then "cold" water goes back to the heater and closes a circuit. Again, the closed loop dead time between the cooler output and cooler input, $\tau = \tau_{CH} + \tau_H + \tau_{HC}$, is long enough so that $\vartheta_{CI}(t)$ and $\vartheta_{C0}(t)$ become almost constant.

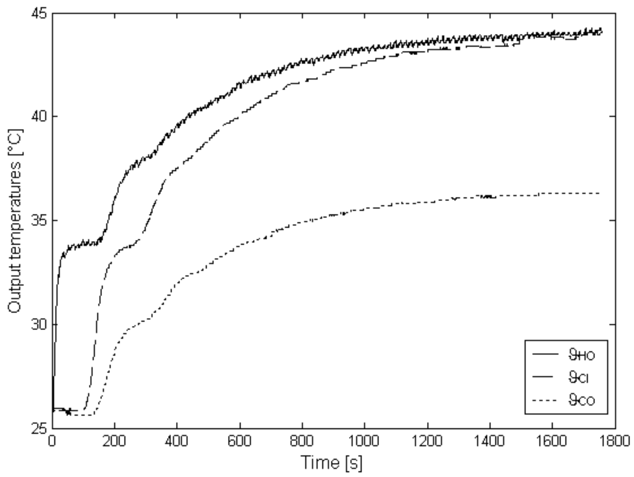


Fig.3 – Heater power step change responses

## 5.1 Estimation of the mass flow rate and the heat transmission coefficient of the heater

There were made no attempts to measure the mass flow rate of water by determination of the diameter of the pipeline, the water-flow velocity, etc. Steady state data in Table 1 can be used for evaluation of $\dot{m}(t)$ by taking into account the fact that more than one steady state can be usually found in a step response of the system, see Fig.3.

The steady state of (1) reads

$$0 = P_0 + c\dot{m}_0\left[\vartheta_{HI0} - \vartheta_{HO0}\right] - K_{H0}\left[\frac{\vartheta_{HO0} + \vartheta_{HI0}}{2} - \vartheta_A\right] \quad (14)$$

i.e. the derivative is assumed identically zero. Table 2 contains the "first" steady state values of temperatures $\vartheta_{HO}(t)$ and $\vartheta_{HI}(t) = \vartheta_{CO}(t - \tau_{CH})$ (at time range cca 150-170s). These data together with data from Table 1 enable to estimate $\dot{m}$ and $K_H$ for a particular setting of input values by inserting these data into (14). The final values of $\dot{m}$ and $K_H$ are taken as the arithmetical mean of all calculated values from these tables for a particular (same) setting. From these values there can be estimated unknown parameters of $\dot{m}$ and $K_H$ in (3) and (7).

Table 1 - Measurements of steady-state temperatures for $u_C = 3\text{V}$

| $u_P$ [V] | $P$ [W] | $\vartheta_{HO0}$ [°C] | $\vartheta_{CI0}$ [°C] | $\vartheta_{CO0}$ [°C] | $\vartheta_A$ [°C] |
|---|---|---|---|---|---|
| 4 | 225 | 38.1 | 38.0 | 31.3 | 22 |
| 4 | 225 | 41.8 | 41.5 | 35.1 | 26 |
| 5 | 225 | 39.4 | 39.3 | 32.9 | 25 |
| 5 | 225 | 40.9 | 40.7 | 34.5 | 27 |
| 6 | 225 | 39.5 | 39.3 | 32.9 | 25.5 |
| 6 | 225 | 38.0 | 37.9 | 33.0 | 23.5 |
| 4 | 300 | 43.5 | 43.2 | 34.9 | 25 |
| 4 | 300 | 42.6 | 42.5 | 33.7 | 23 |
| 5 | 300 | 41.9 | 41.8 | 33.3 | 25.5 |
| 5 | 300 | 44.1 | 43.8 | 36.0 | 25 |
| 6 | 300 | 43.3 | 42.8 | 35.2 | 24 |
| 6 | 300 | 43.4 | 43.1 | 35.3 | 24 |
| 4 | 375 | 48.1 | 47.9 | 37.1 | 24 |
| 4 | 375 | 47.8 | 47.3 | 36.8 | 23.5 |
| 5 | 375 | 48.8 | 48.5 | 38.7 | 25.5 |
| 5 | 375 | 49.9 | 49.7 | 40.0 | 26 |
| 6 | 375 | 48.2 | 47.8 | 38.3 | 23 |
| 6 | 375 | 49.1 | 48.9 | 39.5 | 26.5 |

Table 2 - Measurements of "quasi" steady-state temperatures for $u_C = 3\text{V}$

| $u_P$ [V] | $P$ [W] | $\vartheta_{HO0}$ [°C] | $\vartheta_{HI0}$ [°C] | $\vartheta_A$ [°C] |
|---|---|---|---|---|
| 4 | 225 | 28.8 | 21.7 | 22 |
| 4 | 225 | 33.0 | 26.1 | 26 |
| 5 | 225 | 31.2 | 24.7 | 25 |
| 5 | 225 | 33.8 | 26.9 | 27 |
| 6 | 225 | 31.8 | 25.6 | 25.5 |
| 6 | 225 | 29.6 | 23.1 | 23.5 |
| 4 | 300 | 33.9 | 24.5 | 25 |
| 4 | 300 | 30.7 | 21.7 | 23 |
| 5 | 300 | 33.9 | 25.4 | 25.5 |
| 5 | 300 | 33.9 | 25.1 | 25 |
| 6 | 300 | 32.1 | 23.6 | 24 |
| 6 | 300 | 32.7 | 24.1 | 24 |
| 4 | 375 | 35.5 | 24.1 | 24 |
| 4 | 375 | 35.3 | 23.6 | 23.5 |
| 5 | 375 | 36.4 | 25.2 | 25.5 |
| 5 | 375 | 36.7 | 25.7 | 26 |
| 6 | 375 | 29.2 | 22.9 | 23 |
| 6 | 375 | 32.8 | 26.5 | 26.5 |

The evaluation of these data results in the following numeric estimation (made in MS Excel)

$$\begin{aligned} &p_0 = 5.077\ 10^{-3},\ p_1 = 0.266,\ p_2 = 0.274 \\ &h_0 = 4.1927,\ h_1 = -0.0017,\ h_2 = -15008, \\ &h_3 = -130020,\ h_4 = 716.468,\ h_5 = 77.7722 \end{aligned} \quad (15)$$

where the water density was chosen as $\rho = 993$ kg m$^{-3}$, and $c$ = 4180 J kg$^{-1}$ K$^{-1}$.

The important disadvantage of these estimations is the fact that the results are strongly sensitive to the measurement of the ambient air temperature.

### 5.2 Estimation of the heat transmission coefficient of the coiled insulated pipeline

Data in Table 1 together with the static equation obtained from (4) can be also used for the evaluation of the (constant) heat transmission coefficient $K_P$. Thus, from (4) we have

$$0 = c\dot{m}_0\left[\vartheta_{HO0} - \vartheta_{CI0}\right] - K_P\left[\frac{\vartheta_{CI0} + \vartheta_{HO0}}{2} - \vartheta_A\right] \qquad (16)$$

The final value of $K_P$ is taken as the arithmetical mean again as $K_P = 0.39$ W K$^{-1}$. Obviously, the pipeline is insulated very well and this coefficient does not affect the system dynamics significantly. The measurement is sensitive to $\vartheta_A$ again, and the converter resolution (cca 0.1 °C) disables to find an more accurate value of $K_P$.

### 5.3 Estimation of the heat transmission coefficient of the cooler

A steady state yields (5) of the form

$$0 = c\dot{m}_0\left[\vartheta_{CI0} - \vartheta_{CO0}\right] - K_{C0}\left[\frac{\vartheta_{CO0} + \vartheta_{CI0}}{2} - \vartheta_A\right] \qquad (17)$$

This equation together with data in Table 3 gives the estimation of $K_C$ for a particular setting of $u_C$, similarly as in Section 5.1.

Table 1 - Measurements of steady-state temperatures for various $u_C$. $P = 300\,\mathrm{W}$, $u_P = 5\,\mathrm{V}$.

| $u_C$ [V] | $\vartheta_{HO0}$ [°C] | $\vartheta_{CI0}$ [°C] | $\vartheta_{CO0}$ [°C] | $\vartheta_A$ [°C] |
|---|---|---|---|---|
| 1 | 48.1 | 47.9 | 40.0 | 24 |
| 1 | 45.3 | 45.0 | 36.2 | 21.5 |
| 1 | 46.5 | 46.3 | 38.2 | 25 |
| 2 | 43.3 | 42.9 | 34.7 | 22.5 |
| 2 | 43.3 | 42.8 | 34.9 | 23.5 |
| 2 | 44.5 | 44.3 | 35.8 | 23 |
| 4 | 39.8 | 39.3 | 30.0 | 20.5 |
| 4 | 42.3 | 42.2 | 23.7 | 23 |
| 4 | 43.1 | 42.8 | 34.5 | 25.5 |
| 5 | 39.6 | 39.3 | 31.0 | 21 |
| 5 | 39.9 | 39.6 | 31.6 | 22 |
| 5 | 40.9 | 40.6 | 32.3 | 24 |
| 6 | 40.6 | 40.5 | 32.2 | 23 |
| 6 | 41.1 | 40.9 | 32.6 | 24.5 |
| 6 | 38.6 | 38.4 | 30.2 | 21 |

Note: Temperature values for $u_C = 3$ V are omitted in Table 3 since they can be obtained from Table 1.

The arithmetical mean of particular values of $K_C$ with help of the numerical optimization (MS Excel) give

$$c_0 = 11.8,\ c_1 = 2.755,\ c_2 = -0.19 \qquad (18)$$

### 5.4 Estimation of delays

Delays were estimated graphically from dynamic data (step responses) for appropriate system input changes. The delay of the control action of the heat exchanger (cooler), $\tau_{KC}$, was obtained from the cooling curve (not displayed here due to the limit space). Results are dependent on the particular mass flow rate; hence, arithmetical mean was taken in the final (i.e. for $u_P = 5\,\mathrm{V}$).

$$\tau_H = 3\mathrm{s},\ \tau_{HC} = 110\mathrm{s},\ \tau_C = 25\mathrm{s},\ \tau_{KC} = 12\mathrm{s},\ \tau_{CH} = 7\mathrm{s} \qquad (19)$$

### 5.5 Estimation of masses

Overall masses of water in the heater, the cooler and in the long pipeline were estimated graphically from dynamic characteristics, so that measured and calculated model give a good agreement. Results are the following

$$M_H = 0.12\,\mathrm{kg},\ M_P = 0.2\ \mathrm{kg},\ M_C = 0.3\ \mathrm{kg} \qquad (20)$$

The final comparison of measured step responses and the calculated ones are depicted in Fig.4.

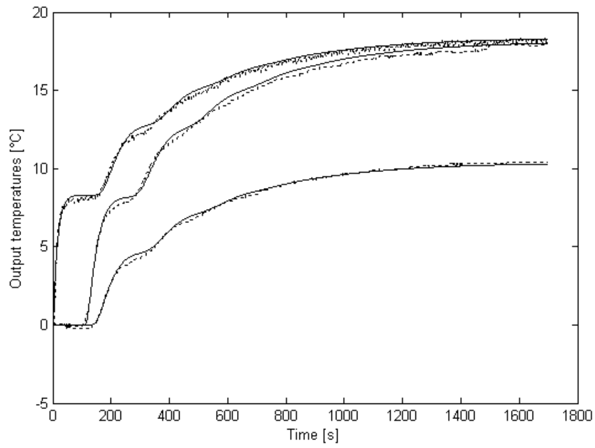


Fig.4 – Comparison of measured (dotted) and calculated (solid) step responses for the settings $u_P = 5\,\mathrm{V}$, $u_C = 3\,\mathrm{V}$, $\Delta P = 300\,\mathrm{W}$, on/off fan is on

## 6 Conclusions

The presented contribution studied a laboratory thermal heating model. The aim was to find a mathematical model of the appliance. In order to avoid needlessly complicated description, we utilized anisochronic modelling philosophy comprising delays as an important factor in the plant dynamics. Unknown parameters were further estimated experimentally. The final graphical comparison of the step responses records a very good agreement of measured and calculated data.

The final results will be used for the verification of control algebraic algorithms developed by authors, in the future research.

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