Comparative Methods for the Polynomial Isolation

CĂLIN ALEXE MUREŞAN,
Department of Mathematics, Faculty of Letters and Science,
University Petroleum and Gas of Ploiesti,
B-dul Bucuresti, No.139, Ploiesti, Prahova,
ROMANIA
acmuresan@upg-ploiesti.ro, c1a2muresan@upg-ploiesti.ro,

Abstract:
We make a comparative study, for the integer and rational polynomial, between the complexity of isolation, with Sturm Sequence and Bisection and Continuity. We will obtain the cost of isolation, of the polynomial roots, which consists in the number of arithmetical operations counted when we use bits representation.

Key-Words: Pre-isolating and isolating the roots, Sturm Sequence, Bisection and Continuity

1 Introduction
The aim of this article is to give our results regarding the “polynomial isolation” for integer and rational polynomials. These results are necessary in pre-isolating respectively isolating the complex roots of a polynomial. We obtain the evaluation of the necessary costs and we discuss about the analytical and empirical efficiency comparing the following methods: Sturm Sequence and Bisection and Continuity.

In this section we introduce the basis notations and notions and present the preliminary results in this field. We denote a real polynomial with

\[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \quad n \geq 1, \]

and its roots with \( x_1, \ldots, x_n \).

Definition 1.1

a) A pre-isolating/isolating interval for a real polynomial represent an open interval \((a, b)\), having as limits two rational numbers, between which there is at most/precisely, one real root of the polynomial.

For an isolating interval we have:

\[ \big\{ (\exists) c \in \mathbb{R} : (x_1, x_2, \ldots, x_n) \in (a, b) \big\} = \{ x_i \} . \]

b) Pre-isolating/isolating the real roots consists in finding for all the polynomial’s roots, disjunctive pre-isolating/isolating intervals.

Definition 1.2

a) Starting from a single interval, we will call “splitting”: the division of it in two intervals equally by length. By repeating the process, where “\( k \)” is an natural number, if we split each of the \( 2^k \) intervals obtained by division, we will obtain \( 2^{k+1} \) intervals and \( 2^{k+1} + 1 \) points, which will be the extremities of the determined intervals; the process will also be called splitting.

b) The recurrent process of a finite number of times “\( k \)”, with “\( k \)” natural number will be called successive splitting or continuous division.

c) A splitting is counted as a single arithmetical operation.

Definition 1.3

For a given function \( g(x) : \mathbb{R} \rightarrow \mathbb{R} \), we denote by \( O(g(x)) \) the set of functions:

\[ O(g(x)) = \{ f(x) : \mathbb{R} \rightarrow \mathbb{R}, (\exists) c, 0 \leq f(x) \leq c \cdot g(x), (\forall) x \geq x_0 \} . \]

In this case for every \( f(x) \) we denote:

\[ O(g(x)) = f(x) . \]

We are saying that “ \( f \) grows at the same rate or it may grow more slowly than \( g \) when \( x \) is very large”. For more details see [1] and [2].

Definition 1.4

Let be \( P(x) \in \mathbb{C}[x] \). We define:

a) \( \text{sep}(P) = \min \{ | x_i - x_j | / x_i \neq x_j, \ 1 \leq i, j \leq n \} \) the minimum roots separations.

b) \( \| P \| = \sqrt{|a_0|^2 + |a_1|^2 + \ldots + |a_n|^2} \)

c) \( \| P_\infty \| = \max \{ |a_0|, |a_1|, \ldots, |a_n| \} \),

d) The discriminant of a polynomial \( P \in \mathbb{C}[x] \) with leading coefficient \( a_n \) and the roots \( x_1, \ldots, x_n \) is defined as:

\[ \text{disc}(P) = a_n^{2n-2} \cdot \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 . \]

e) The Mahler Measure of the polynomial \( P \), denoted by \( M(P) \) is:
\[ M(P) = M[P(x)] = a_n \prod_{j=1}^{n} \max \{|l_j| \}. \]

**Definition 1.5**

For \( P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \); \( a_i \in \mathbb{R}; i = 0, \ldots, n \), \( a_n \neq 0 \):

- \( f_0(x) = P(x); f_1(x) = -P'(x); f_2(x) = -f_0(x) \) \mod \( f_1(x) \)
- \( \ldots; f_{i+1}(x) = -f_{i-1}(x) \) \mod \( f_{i}(x) \);
- \( \ldots; \) until \( f_{p+1}(x) = 0, p \in \mathbb{N} \). The series \( \{f_0, f_1, \ldots, f_p\} \) is called Sturm's series attached to the polynomial \( P(x) \).

**Theorem 1.1 Sturm's Theorem.**

For polynomial \( P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \); \( a_i \in \mathbb{R}; i = 0, \ldots, n \), \( a_n \neq 0 \) and for \( a, b \in \mathbb{R} \) with \( a < b \); if \( \{f_0, f_1, \ldots, f_p\} \) is Sturm's series attached to the polynomial \( P(x) \) and if \( P(a) \neq 0 \) and \( P(b) \neq 0 \), then the real number of roots of \( P \), from the interval \([a, b]\), will be: \( V(a) - V(b) \).

**Proof:** see [3] from References.

### 2. The Roots Pre-isolation

**Theorem 2.1** [4, 5] For an arbitrary complex polynomial, \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \); with \( n \geq 1, a_n \neq 0 \) and for \( P \in \{1, 2, \ldots, n\} \) such that \( x_1 \geq x_2 \geq \ldots \geq x_p \geq 1 \geq x_{p+1} \geq \ldots \geq x_n \) then:

- a) \( 1 \leq |x_i| \leq \left( \frac{M(P)}{|a_n|} \right)^{1/i} \) \leq \left( \frac{|P|}{|a_n|} \right)^{1/i} \) for \( i = 1, p \) and
  \[ \left( \frac{|a_0|}{|P|} \right)^{1/n-i+1} \leq \left( \frac{|a_0|}{M(P)} \right)^{1/n-i+1} \leq |x_i| \leq 1, \]
  for \( i = p+1, n \).
- b) If \( 1 > |x_i| > \ldots > |x_p| \), then \( M(P) = a_n \) for \( i = 1, n \):
  \[ \left( \frac{|a_0|}{|P|} \right)^{1/n-i+1} \leq \left( \frac{|a_0|}{M(P)} \right)^{1/n-i+1} \]
  \[ \leq |x_i| < 1 = \left( \frac{M(P)}{|a_n|} \right)^{1/i}. \]

**Corollary 2.1**

Let be \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \), \( n \geq 1 \). \( P \in \mathbb{R}[x] \), then \( \exists r, R \) real, such that

\[ r \leq |x_i| \leq R, (\forall) i = 1, n; \]

i) \( R = \frac{M(P)}{|a_n|}, R = \frac{|P|}{|a_n|} > 1 \) and

\[ r = \frac{|a_0|}{M(P)} \]

ii) \( R = \frac{L(P)}{|a_n|} > 1 \) and \( r = \frac{|a_0|}{L(P)} \).

**Theorem 2.2 Cauchy** [6] Be it \( P(x) \in \mathbb{R}[x] \)

\( P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + x^n \); \( a_i \in \mathbb{R}; i = 0, n-1, n \geq 1 \). Be it the sequence: \( a_1 < 0, a_2 < 0, \ldots, a_k < 0, \) \( i_1 < i_2 < \ldots < i_k \), for the negatives coefficients of polynomial where \( k \) is the number of the negatives coefficients and be it:

\[ R = \max_{1 \leq j \leq k} \left\{ \left( \frac{k \cdot a_{i_j}}{a_{n-j}} \right)^{1/n-j} \right\}. \]

Then, each of the real zeros of the polynomial, satisfy the relation: \( x \leq R \).

**Corollary 2.2 Cauchy** In condition to the previous theorem if:

\( P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} + a_n x^n \),

then, each of the real (positive) zeros of the polynomial, satisfy the relation \( x \leq B \) where:

\[ R = \max_{1 \leq j \leq n, a_{i_j} < 0} \left\{ \left( \frac{k \cdot a_{i_j}}{a_{n-j}} \right)^{n-j} \right\}. \]

Demonstration is immediately and is based on the previous results. For others bounds see [3, 7].

**Theorem 2.3** [4] For \( n \geq 1, a_n \neq 0 \), \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \in \mathbb{R}[x], \)

if exist \( P \in \{1, 2, \ldots, n\} \) such that \( x_1 \geq x_2 \geq \ldots \geq x_p \geq 1 \geq x_{p+1} \geq \ldots \geq x_n \) then:

- a) \( \text{sep}(P) \geq \frac{\sqrt{\text{disc}(P)}}{a_n^{n-1} n/2 (n-1)} \left( \frac{|a_n|}{|P|} \right)^{p+1/2} \) \( (n-1) \)
- b) If \( 1 > |x_i| > \ldots > |x_p| \geq |x_{p+1}| \geq \ldots \geq |x_n| \) then
polynomials of the sequence. The summ of the degrees of the polynomials is "n". Then the cost, without the interpretation of the rational number for the evaluation, using Horner method, is $O\left( n^2 \right)$.

**Proposition 3.1**

If $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$; $n \geq 1$, $a_n \neq 0$, $P(x) \in \mathbb{Q}(x)$, the cost of $V(x)$ evaluation from Sturm’s Sequence in the rational point: $x = \frac{a}{b}$, $a \neq 0$, $b > 0$ is:

$$O\left( n^2 \cdot \max \{ \tau, \log_2 |a/b| \} \right),$$

where $\tau \leq \log_2 L(P)$, is the maximum coefficient bit size of the polynomial.

We can follow: [10], [11], [12].

**Observation 3.2**

As we can see, the complexity of the isolation depends on the nominator and denominator of the isolation interval’s limits. The evaluation of this number or for the numbers of the bits involved is necessary here or in other isolation proceed. We can follow now the examples:

In paper [13] is used, in „Theorem 6.6“ the inequality:

$$2 \cdot L(P) \geq d, d = \max \{ |a|, |b|, e, f \},$$

(5)

each interval having as limits the rational numbers, \( \frac{a}{b}, \frac{e}{f} \) in which we evaluate the Sturm’s Sequence sign.

In paper [14] is used, in „Theorem 3“ the inequality:

$$2 \cdot L(P) \geq \left| \frac{d}{b} \right|,$$

(6)

for each rational number \( \frac{a}{b} \) in which we evaluate the Derivatives Sequence.

The classification of the complexity of the previous rational numbers, representation denoted by \( d \), in function of the bit size \( d_B \), \( d \leq 2^{d_B} \) we can find it in [3]:

$$d_B = 1;$$

(7)

which corresponds to the case of a small number of widely-separated roots;

$$d_B = O\left( \log_2 \left( n \cdot \| P \| \right) \right),$$

(8)

which corresponds to a large number of roots, but averagely” spaced;

$$O\left( n \cdot \log_2 \left( n \cdot \| P \| \right) \right) = \log_2 (sep(P));$$

(9)

which corresponds to the worst case of roots as close together as possible

In [12] in Section 3. Average-case...
complexity, Lemma 2, the average bit size of a isolation point is:

\[ O(n + \tau) \]  

(10)

where \( \tau \leq \log_2 (L(P)) \), is the maximum coefficient bit size of the polynomial.

These considerations make us able to state the following definition.

**Definition 3.1.** "The worst case of roots" for isolation, will be that in which the representation of the limits of the pre-isolation intervals, are rational numbers \( d = \frac{a}{b} \) having the bits representation:

\[ d_B = O\left(n \cdot \log_2 n \cdot \| P \| \right). \]

**Theorem 3.1.** For isolating the real roots of the polynomial \( P(x) \in \mathbb{R}[x] \),

\[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0, \quad i = 0, n, \quad n \geq 1, \quad a_n \neq 0, \]

the complexity of the evaluation of the Sturm's Sequence in "the worst case of roots" in function of the operation counted on bits is:

\[ O\left(n^4 \cdot \log_2^2 \left( n \cdot \| P \| \right) \right). \]

**Demonstration:** From Corollary 2.3 for the roots pre-isolation, we need \( O\left(n \cdot \log_2 \left(n \cdot \| P \| \right) \right) \) operations of continuous divisions.

From **Proposition 3.1**

\[ O\left(n^2 \cdot \max \{ \log_2 L(P), \log_2 |a|, b \} \right) \]

is the order of the cost of the evaluation of Sturm's Sequence into a rational point.

(12)

Now using **Definition 3.1**

\[ \log_2 |a|, b = O\left(n \cdot \log_2 n \cdot \| P \| \right) \]

and from **Suposition 3.1** then from (11) and (12) we obtain the cost for all the evaluation of the Sturm's Sequence involved in isolation.

**Observation 3.3** The isolation is dominated from the previous cost, we can observe [12], [3], or, **Suposition 3.2** and is one of the simplest complexity and is in concordance with the empirical results as we can see in [15].

**Observation 3.4** For the polynomial:

\[ P(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{R}[x], \quad n \geq 1, \quad a_0 \cdot a_n \neq 0. \]

the complexity of the evaluation of the Sturm's Sequence in the real roots isolation proceed is:

\[ O\left(n^4 \cdot \log_2^2 \left( n \cdot \| m \cdot \| P \| \right) \right), \]

where "\( m > 0 \)" is the least common multiple of the denominator coefficients of \( P \), \( \{a_0, a_1, a_2, \ldots, a_n\} \).

**Demonstration:**

The least common multiple of the denominator coefficients of \( P \), \( \{a_0, a_1, a_2, \ldots, a_n\} \) using \( O(n^2) \) arithmetical operations. Be it \( R(x) = m \cdot P(x), \quad R(x) \in \mathbb{R}[x] \). Now we have \( \| R \| = m \cdot \| P \| \), and from **Observation 3.3** we obtain the result.

### 3.2 The Bisection and Continuity Technique

The continuous division is repeated until the isolation of the roots is accomplished. We use the polynomial continuity for each interval obtained from successive divisions, in order to know if roots exist in the pre-isolation intervals. The algorithm costs are dominated by the complexity of the polynomial’s evaluations, to the limits of the intervals. We find in paper, similar results, see [16].

**Proposition 3.2** Be it the polynomial \( P(x) \), square free, with the degree "\( n \geq 1 \)". The sign evaluation for the polynomial in the point

\[ x = \frac{a}{b} \in \mathbb{Q}, \quad a \neq 0, \quad b > 0 \]

in "the worst case of roots" in function of the operation counted on bits is:

\[ O\left(n^2 \cdot \log_2 \left( n \cdot \| P \| \right) \right). \]

**Demonstration:** With Horner method in polynomial evaluation we need \( O(n) \) operation of multiplications. Each of the numbers \( x = \frac{a}{b} \in \mathbb{Q}, \quad a \neq 0, \quad b > 0 \) need

\[ d_B = \log_2 |a|, b = O\left(\log_2 \left( n \cdot \| P \| \right) \right) \]

bits, as we can see in **Definition 3.1**. Counting the numbers of operations between the bits we obtain the result.

**Theorem 3.2** Be it the polynomial \( P(x) \), square free, with the degree "\( n \geq 1 \)",

\[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0, \]

\( a_n \neq 0 \) and \( a_0 \neq 0 \), then the cost of the real roots isolation in function of the operation counted on bits have the order:

\[ O\left(n^3 \log_2^2 \left( n \cdot \| P \| \right) \right). \]

**Demonstration:** From Corollary 2.3 for roots pre-isolation we need the number of the divisions with the order:

\[ f(n) = O\left(n \log_2 n \cdot \| P \| \right) \]

operations. (13)

Based on **Suposition 3.1** the cost of the entire sign evaluation for the polynomial is:

\[ O\left(n^2 \log_2 \left( n \cdot \| P \| \right) \right) \cdot O\left(\log_2 \left( n \cdot \| P \| \right) \right) = \]

\[ O\left(n^3 \log_2^2 \left( n \cdot \| P \| \right) \right), \quad \text{ (14)} \]

These complexity dominate the isolation, see **Suposition 3.2** or [16].
Observation 3.5 Similar with previous observation for the real roots isolation, of the polynomial, \( P(x)=a_nx^n+...+a_1x+a_0 \in \mathbb{R}[x] \), \( n \geq 1 \), \( a_0 \cdot a \neq 0 \), square free, the complexity have the order: 
\[
O\left[n^3 \cdot \log_2^2 \left( n \cdot m \| \! \| P \| \| \right)\right],
\]
where ‘\( m>0 \)’ is the least common multiple of the denominator coefficients of \( P \).

Observation 3.6

The previous cost dominates the isolation complexity as we can see in the papers [16], [17] or Supposition 3.2, and is one of the smallest complexity.

Observation 3.7

Comparing the results of the isolation with the previous methods, we can put the question if the method of Bisection and Continuity is really better than the method of Sturm’s Sequence.

Both evaluations will use the pre-isolation cost but in fact, the number of the intervals needed in isolation with Sturm’s Sequence is smaller than the number involved in Bisection and Continuity. That is the reason why the methods are competitive in practice.

References:


