Abstract: The main goal of this paper is to develop numerical procedures for analyzing the swirling flows with helical vortex breakdown subject to various boundary conditions imposed by the helical modes. For non axisymmetrical case a boundary adapted spectral collocation technique was implemented. For modes $m = 0$, $m = 1$ and $m = -1$, the eigenvalue problem governing the stability satisfy sophisticated boundary conditions. For our numerical evaluation we develop a modified tau method based on orthogonal expansions with shifted Chebyshev polynomials, searching the numerical approximation of the unknown perturbation field directly in the physical space. Both methods provide an acceptably accurate approximation of the spectrum without any scale resolution restriction and relevant information on perturbation amplitudes for stable or unstable modes, the maximum amplitude of the most unstable mode and the critical distance where the perturbation is the most amplified.

Key-Words: swirling flow, trailing vortex, spectral collocation technique, tau method.

1 Introduction
In practical engineering problems such as control of high-Reynolds number flow [1-4], stability analyses are needed to predict vortex motion and effects that vortices can produce. The investigations concerned the values of parameters for which the vortex become unstable may imply a large amount of measurement, thus one must resort to numerical techniques.

The present paper is focused on developing an analytical and numerical technique for analyzing the eigenvalue problem governing the linear stability of
an inviscid swirling fluid flow under small perturbations. Due to the lack of spectral theory with respect to non-selfadjoint differential operators this type of problems are far from being solved.

We developed in this paper a comparative numerical investigation upon stability of a trailing vortex, subject to various boundary conditions imposed by the helical modes. The paper is organized as follows. The eigenvalue problem governing the linear stability analysis for inviscid swirling flows against normal mode perturbations is defined in Section 2. The third section of the paper propose a new radial spectral approximation for the eigenvalue problem. The numerical investigations are applied upon the model of a trailing vortex. The main results of the paper are summarized in Section 5.

2 Problem Formulation

Assuming a steady columnar flow the velocity profile is written as:

\[ \mathbf{V}(r) = [U(r), V(r), W(r)] \]

(1)

where \( U \) represents the axial velocity component, \( V \) the radial velocity component and \( W \) the tangential component of the velocity all depending only on radius. We use the following flow fields decomposition: velocity \( \mathbf{V} = \mathbf{V} + \mathbf{v} \), pressure \( p + \pi \) where \( (\mathbf{V}, p) \) is the base flow, and \( (\mathbf{v}, \pi) \) is the perturbation considered small. Since the base flow obey the Navier-Stokes equations, the evolution of such small perturbations of the basic flow is governed also by the following dimensionless linearized Navier-Stokes equations

\[
\nabla \cdot \mathbf{v} = 0, \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{v} = -\nabla \pi + \frac{1}{Re} \Delta \mathbf{v} \tag{2}
\]

For high Reynolds numbers a restrictive hypothesis to neglect viscosity can be imposed. We consider the following factorization of the small perturbations:

\[
[u(t, z, r, \theta), v(t, z, r, \theta)] = [IF(\tau, r), G(\tau, r), H(\tau, r), P(\tau)] e^{i(k_z + m \theta - \omega t)} \tag{3}
\]

Introducing the factorization form (3) into the linearized Navier-Stokes equations (2) we obtain the following system of first order differential equations, used also in [1]

\[
G' + \frac{G}{r} + \frac{mH}{r} + kF = 0 \tag{4a}
\]

\[
iVG' + \left[ \omega + iV' - \frac{mW}{r} - kU \right] G - \frac{2WH}{r} + P' = 0 \tag{4b}
\]

\[
VH' + \left[ -i\omega + \frac{imW}{r} + ikU + V \right] H + \left[ iW + \frac{iW}{r} \right] G + \frac{imP}{r} = 0 \tag{4c}
\]

\[
VF' + \left[ -i\omega + \frac{imW}{r} + ikU + U' \right] F + iU' G + ikP = 0 \tag{4d}
\]

where \( \theta \) denotes differentiation with respect to the radius. The axial singularities in a cylindrical polar coordinates system occur due to the presence of terms \( 1/r, r \)-radial distance. This terms occur in the stability governing equations of the swirl flow because the computational domain is set to \([0, r_{max}]\) and certain boundary conditions must be specified in \( r = 0 \). In this case, \( \lim_{r \to 0} \frac{\partial V}{\partial \theta} = 0 \) and \( \lim_{r \to 0} \frac{\partial \pi}{\partial \theta} = 0 \).

The base flow is independent of \( \theta \) and this tidies up to \( \lim_{r \to 0} \frac{\partial V}{\partial \theta} = 0, \lim_{r \to 0} \frac{\partial \pi}{\partial \theta} = 0 \). Representing the perturbation velocity field by \( \mathbf{v} = (v_z, v_r, v_\theta) \) we have

\[
\lim_{r \to 0} \frac{\partial v_z}{\partial \theta} = \lim_{r \to 0} \frac{\partial v_r}{\partial \theta} = 0 \quad \text{implies} \quad \lim_{r \to 0} \frac{\partial v_\theta}{\partial \theta} = 0 \quad \lim_{r \to 0} \frac{\partial \pi}{\partial \theta} = 0 \quad \lim_{r \to 0} \frac{\partial \pi}{\partial \theta} = 0 \quad \lim_{r \to 0} \frac{\partial \pi}{\partial \theta} = 0 .
\]

In order for these equalities to hold, each component of the resultant vector must be zero. Summarizing, we have

\[
mG + H = 0, \quad G + mH = 0, \quad mF = 0, \quad mP = 0 , \quad \text{in the limit along the centerline} \quad r = 0 . \quad \text{The homogenous first order differential system (4) is completed with the following boundary conditions in axis origin:}
\]

\[
(l m > 1), \quad F = G = H = P = 0, \tag{5a}
\]

\[
(m = 0), \quad G = H = 0, F, P \quad \text{finite}, \tag{5b}
\]

\[
(m = \pm 1), \quad H \pm G = 0, F = P = 0 . \tag{5c}
\]

For a truncated radius distance \( r_{max} \) selected large enough such that the numerical results do not depend on that truncation of infinity the boundary conditions read

\[
(l m > 1), \quad F = G = H = P = 0 \tag{6a}
\]
The relations with to infinity, we developed a spectral numerical procedure. The key issue here is the choice of the boundary conditions. To investigate the cases \(|m|>1\), when the system (4a)-(4d) obey Dirichlet boundary conditions at axis and to infinity, we developed a spectral numerical procedure. The key issue here is the choice of the grid and the choice of the modal trial basis based on orthogonal expansion functions satisfying the boundary conditions.

Following [6] we define the boundary-adapted basis functions \(\{\phi_k\}, k=1,\ldots,N\) of modal type

\[
\phi_k(r) = \begin{cases} 1 - \frac{r}{r_{\text{max}}}, & k = 1, \ldots, N-1 \\ \frac{1}{\sqrt{2(2k+1)}} \left[ T^*_k(\xi) - T^*_k(1) \right], & k = N 
\end{cases}
\]

with \(T^*_k\) the shifted Chebyshev polynomials on \([0,r_{\text{max}}]\). Each function provides one particular pattern of oscillation and optimize the interpolative procedure. Each of the basis functions from (7) meet the relations

\[
\phi_1(\eta = 0) = 1, \quad \phi_1(r_N = r_{\text{max}}) = 0, \\
\phi_N(\eta = 0) = 0, \quad \phi_N(r_N = r_{\text{max}}) = 1, \\
\phi_k(\eta = 0) = \phi_k(r_N = r_{\text{max}}) = 0, \\
\phi_k(r_j) \neq 0, \quad j = 1..N, \quad k = 2..N-1
\]

which implies that each functions \(F,G,H,P\) satisfy the Dirichlet boundary conditions.

In the method applied here, the numerical approximation \(v_N\) of the unknown perturbation field \(v\) is searched in a Hilbert subspace of algebraic polynomials of degree \(N\), such that the equation is satisfied in a certain number of collocation points \(x_i\) on \([0, r_{\text{max}}]\).

The proposed method allowed us to discard the first and last collocation nodes, expansion functions satisfying the boundary conditions from the construction of our modal boundary-adapted basis. In this way the critical singularities which occurred in evaluating terms like \(1/r\) for the numerical treatment of the eigenvalue problem were eliminated. The solution is approximated with respect to this expansion set of functions,

\[
(F,G,H,P) = \sum_{\ell=1}^{N} (f_\ell, g_\ell, h_\ell, p_\ell) \eta_\ell(r)
\]

A modified Chebyshev Gauss grid \(\Xi = \{\xi_j\}_{0 \leq j \leq N-1} \subset [-1,1]\) was constructed

\[
\xi_j = \cos \left( \frac{j\pi}{N-1} \right), \quad j = 0 \ldots N-1
\]

mapped into the physical range of our problem by a linear transformation.

The collocation nodes clustered near the boundaries diminishing the negative effects of the Runge phenomenon. Another aspect is that the convergence of the interpolation function on the clustered grid towards unknown solution is extremely fast.

Let us denote by \(r = \text{diag}(\eta), \frac{1}{r} = \text{diag}(1/\eta)\), \(i = 0,1,\ldots, N-1\), \(\{\phi_{ij}\}_{i=0}^{N-1}, \phi_{ij} = \phi_j(\eta_i)\) \(\{\eta_i\}_{i=0}^{N-1}\),

\[
[U] = \text{diag} (U(\eta)), [V] = \text{diag} (V(\eta)), [W] = \text{diag} (W(\eta)), 1 \leq i \leq N
\]

The eigenvalue problem governing the inviscid stability analysis has now the computational form:

\[
Dg + \left[ \frac{1}{r} \right] \phi g + m \left[ \frac{1}{r} \right] \phi f + k \left[ \phi \right] F = 0
\]

\[
i[V] Dg + \omega \phi g + i[V] \phi f - m \left[ \frac{W}{r} \right] \phi g - k \left[ \phi \right] F + \left[ U \right] \phi g + D p = 0
\]

\[
[V] D\tilde{h} - i\omega \phi \tilde{g} + \left[ \frac{W}{r} \right] \phi \tilde{g} + ik \left[ U \right] \phi \tilde{g} + \left[ V \right] \phi \tilde{f} + \left[ U' \right] \phi \tilde{f} + i k \left[ \phi \right] P = 0
\]

\[
[V] D\tilde{f} - i\omega \phi \tilde{f} + \left[ \frac{W}{r} \right] \phi \tilde{f} + i k \left[ U \right] \phi \tilde{f} + \left[ U' \right] \phi \tilde{f} + i k \left[ \phi \right] P = 0
\]

The shifted Chebyshev polynomials that we have used obey the recurrence relation:
$T_n^*(r) = \frac{r_{\text{max}}}{4} \left( \frac{n-1}{r_{\text{max}} - r} \right) \left[ T_{n-1}^*(r) - T_{n+1}^*(r) \right], n \geq 2 \quad (12)$

Based on this, the interpolation derivative matrix $D$ was evaluated by:

$$
D = \begin{bmatrix}
-1 - \frac{1}{10} \left( \frac{16n}{r_{\text{max}}^2} - \frac{8}{r_{\text{max}}} \right) & E_3(\xi) & \ldots & E_{N-1}(\xi)
-1 - \frac{1}{10} \left( \frac{16n}{r_{\text{max}}^2} - \frac{8}{r_{\text{max}}} \right) & E_3(\xi) & \ldots & E_{N-1}(\xi)
\vdots & \vdots & \ddots & \vdots 
-1 - \frac{1}{10} \left( \frac{16n}{r_{\text{max}}^2} - \frac{8}{r_{\text{max}}} \right) & E_3(\xi) & \ldots & E_{N-1}(\xi)
\end{bmatrix}
$$

with

$$
E_k(r) = \frac{1}{\sqrt{2(2k+1)}} \cdot \frac{r_{\text{max}}}{4r_{\text{max}} - r} \cdot \left( [k-2]T_{k-2}^* - (k-2)H_k^* + kT_k^* \right) \quad (13)
$$

For a stabilization of the Gibbs phenomenon a Lanczos type $\sigma$ factor [6] was used,

$$
\sigma_k = \frac{N \sin \frac{2\pi k}{N}}{2\pi k}, 1 \leq k \leq N 
$$

The spatially or temporal stability (classified for open flows as in [9]) under infinitesimal perturbations is reduced to the study of an algebraic eigenvalue problem. The study leads to a dispersion relation connecting in fact the growth rate $\omega$ and the axial wavenumber $k$ as a consequence of the condition that nontrivial eigenvalues to exist.

The exponential structure in (3) allows the exponential decreasing error. In the axial wavenumber $c$ allowed to grow/decay and oscillate in space:

$$
\omega \text{ is given and we obtain } k \text{ from the dynamic equations.}
$$

This modal boundary adapted algorithm allows us to obtain the eigenvalue, the eigenvector, the index of the most unstable mode, the maximum amplitude of the most unstable mode and the critical distance where the perturbation is the most amplified. The main advantages of the proposed method consist in reducing the computational time by reducing the matrices order to $(4N-8)^2$ and for a certain spectral parameter $N$ we obtain an exponential decreasing error.

### 3.2 An Orthogonal Expansion Based Tau Method

For the cases that we have investigate here, the sophisticated boundary conditions need a different numerical approach for each situation.

Following [10] we will turn our difficult eigenvalue problem into a system of linear equations describing the hydrodynamical context for the cases $m = 0$ and $m = \pm 1$.

The difference between the classical tau method and the modified version proposed here is given by the selected spaces involved in the approximation process. We define the perturbation amplitudes as a finite series of Chebyshev polynomials

$$
(f, g, h, p) = \sum_{k=1}^{N} (f_k, g_k, h_k, p_k) \cdot T_k^* \quad (16)
$$

where $T_k^*$ are shifted Chebyshev polynomials on the physical domain $[0, r_{\text{max}}]$. To reduce the system (4a)-(4d) to a finite dimensional problem we multiply each equation with $T_j^* \cdot w(r), j = 1..N-1$ and take the inner product of terms depending on radial variable in the weighted $L^2(0, r_{\text{max}})$ space

$$
(f, g)_w = \int f \cdot g \cdot w dr, \quad w(r) = \left( \frac{4 \cdot \left( \frac{r_{\text{max}}}{2} \right)^2 - 1}{4 \cdot \left( \frac{r_{\text{max}}}{2} \right)^2 - 1} \right)^{-1} \quad (17)
$$

We obtain a set of $4(N-1)$ linear equations. The boundary conditions provide the eight remaining equations. The choice of the shifted polynomials instead of the classical $[-1, 1]$ defined Chebyshev polynomials is motivated by the orthogonality of the shifted class directly in the physical space and is no need for a numerical interpolation of the jacobian.

Let

$$
I_{ij} = (UT_i^*, T_j^*)_w, \quad J_{ij} = (UT_i^*, T_j^*)_w, \\
K_{ij} = (WT_i^*/r, T_j^*)_w, \quad L_{ij} = (WT_i^*/T_j^*)_w, \\
M_{ij} = (WT_i^*, T_j^*)_w, \quad O_{ij} = (WT_i^*, T_j^*)_w, \\
P_{ij} = (UT_i^*, T_j^*)_w.
$$

The first truncated $4(N-1)$ equations are

$$
k \sum_{1}^{N} f_{jk} j + g \cdot j \cdot r_{\text{max}} \cdot c + g2 \cdot \frac{r_{\text{max}}^2}{2} \cdot I_{jj} + \\
+ \sum_{k \text{ odd}}^{N} g_k \frac{2(k-1)}{r_{\text{max}}} \left( \sum_{k \text{ even}}^{N} 2I_{jr} \right) + \\
+ \sum_{k \text{ even}}^{N} g_k \frac{2(k-1)}{r_{\text{max}}} \left( \sum_{k \text{ odd}}^{N} 2I_{jr} + I_{jj} \right) + mh r_{\text{max}} c = 0 \quad (18a)
$$

$$
k \sum_{1}^{N} g_{jk} j - \omega g \cdot r_{\text{max}} \cdot c + m \sum_{1}^{N} g_k K_{jk} + 2 \sum_{1}^{N} h_k K_{jk} = 
$$
for the reason that for $r > 3$ the axial velocity $U$ is essentially 0, as it is shown in [8].

The critical distances $r_c$ where the perturbation is the most amplified are given in Table 1 along with the most amplified axial wavenumber $k$ in the zero-external flow jet case for various modes.

Table 1. The most unstable mode $k$ and the critical distance $r_c$ for the case of zero-external flow jet and the swirl parameter $q = 0.05$ and $\omega = 0.1$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$k$</th>
<th>$r_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0.6411</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1.2865</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1.4762</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1.7003</td>
</tr>
</tbody>
</table>

Fig. 1 The absolute values of the perturbation amplitudes in the case $a = 0$, $q = 0.05$, $\omega = 0.1$, for mode $m = 2$, for $N = 100$ collocation nodes.

In Figure 1 it is shown that the sensitivity on axial perturbations is located near the center, for a small radius.

Considering the equation (11a) as a three parameter set

$$\Pi(m, N, k(N)) = G + \frac{1}{r} \sum_{i=1}^{N} g(r) + \frac{m}{r} \sum_{i=1}^{N} h_i(r) + k \sum_{i=1}^{N} \phi_i(r)$$

the error vector $e_r(\Pi) = \{e_r(R), k = 1..N\}$ is defined as $e_r(\Pi) = D \hat{\Pi} = \sum_{i=1}^{N} \hat{\Pi} + k \hat{\Pi}$. As the spectral parameter $N$ is varying we retain the norm of the error vector $\|e_r(\Pi)\| = \left(\sum_{k=1}^{N} e_r(\Pi)^2\right)^{1/2}$. Figure 2 presents the behavior of the error as the number of collocation nodes is increasing.

In the zero-external-flow swirling jet, we also investigated the evolution of spatial stability characteristics with respect to increasing values of the swirl parameter for the cases with $m = 0$, $m = 1$.
and \( m = -1 \). It is found that the mode \( m = -1 \) is more unstable than its positive counterpart \( m = 1 \) (Fig.3). The dependence of critical frequency on the swirl parameter, for modes \( m = 0 \), \( m = 1 \) and \( m = -1 \) is studied in Fig.4.

![Fig.2 Error vs. spectral parameter N, for the case \( m = 2 \), \( a = 0 \), \( q = 0.05 \), \( \omega = 0 \).](image)

![Fig.3 The neutral curves \((q,-k_i)\) in the case \( m = 0 \), \( m = 1 \) and \( m = -1 \), obtained by tau method, using \( N = 6 \) expansion terms.](image)

![Fig.4 The dependence of critical frequency on the swirl parameter.](image)

5 Conclusion

In this paper we developed numerical procedures to investigate the spatial stability of a trailing vortex, using two different numerical methods.

For non axisymmetrical modes the first method is assessed for the construction of a certain class of orthogonal expansion functions satisfying the boundary conditions. The choice of the grid and the choice of the modal trial basis eliminate the singularities, the scheme based on shifted Chebyshev polynomials providing good results. For modes \( m = 0 \), \( m = 1 \) and \( m = -1 \), the eigenvalue problem and its sophisticated boundary conditions were translated into a linear system using a modified tau method based on orthogonal shifted Chebyshev expansions. The numerical approximation of the unknown perturbation field was searched directly in the physical space. The collocation method is more accurate, however the tau method is less expensive with respect to the numerical implementation costs, i.e. numerical results are obtained for a much smaller number of terms.

References: