A Functional–Integral Equation via Weakly Picard Operators

MARIA DOBRIŢOIU
Department of Mathematics
University of Petroşani
str. Universităţii, 20, 332006, Petroşani
ROMANIA
mariadobritoiu@yahoo.com

ANA-MARIA DOBRIŢOIU
Graduate in Mathematics-Informatics at
Babeş-Bolyai University of Cluj-Napoca
ROMANIA
annad_2009@yahoo.com

Abstract: Using the Picard operators technique and a result given by I.A. Rus in the paper [11], a theorem of data dependence of the solution of the functional–integral equation

\[ x(t) = g(t, x(t), x(a)) + \int_{a}^{t} K(t, s, x(s), x(a))ds, \quad t \in [a, b], \]

where \( K : [a,b] \times R^2 \to R, \ g : [a,b] \times R^2 \to R, \) is given.

Key-Words: Functional–integral equation, solution, weakly Picard operator, data dependence.

1 Introduction

The data dependence is an important part of the study of the solution of various functional-integral equations. This problem has been studied in many papers, of which we mention several treatises: Gh. Coman, I. Rus, G. Pavel, I.A. Rus [1], R. Precup [7], I.A. Rus [8], [12], M.A. Şerban [16] and several articles: G. Dezső [2], V. Mureşan [6], I.A. Rus [10], [11], [13], I.A. Rus and S. Mureşan [9], A. Sintămărian [14], J. Sotomayor [15], M.A. Şerban [17] and others.

Also, we mention the results of data dependence of the solution of some integral equations, obtained by M. Dobriţoiu in the papers [3], [4] and [5].

The purpose of this paper is to give a result which contains the conditions of data dependence of the solution of the functional-integral equation

\[ x(t) = g(t, x(t), x(a)) + \int_{a}^{t} K(t, s, x(s), x(a))ds, \quad t \in [a, b]. \]

In order to establish this result, the Picard operators technique and a theorem given by I.A. Rus in the paper [13] have been used. Also, some results presented by I.A. Rus in the papers [11], [12] and M.A. Şerban in the papers [16], [17] have been useful.

2 Notations and preliminaries

Let \( (X,d) \) be a metric space and \( A : X \to X \) an operator. In this paper we shall use the following notations:

\[ P(X) = \{ Y \subset X / Y \neq \emptyset \}; \]

\[ A^0 = \{ x \}, \ A^1 = \{ A(x) \}, \ A^{n+1} = \{ A^n(x) \}, \ n \in N \] – the iterates of the operator \( A \)

\[ I(A) = \{ Y \in P(X) / A(Y) \subset Y \} \) – the family of nonempty subsets of \( X \), invariant for \( A \)

\[ F_A = \{ x \in X / A(x) = x \} \) – the fixed points set of \( A \)

\[ H(U,V) = \max \{ \sup_{x \in U} d(x,V), \sup_{y \in V} d(y,U) \} \] – Pompeiu-Haussdorf metric.

\[ C[a,b] = \{ f : [a,b] \to R / f \) continuous function} \]

We present below several results from the Picard operators theory and some results given by I.A. Rus in the paper [13], which are useful in order to obtain the main result of this paper.

Definition 1. ([13]) Let \( (X,d) \) be a metric space. An operator \( A : X \to X \) is a Picard operator (PO) if there exists \( x^* \in X \) such that:

(i) \( F_A = \{ x^* \} \)

(ii) the sequence \( \{ A^n(x^*) \}_{n \in N} \) converges to \( x^* \), for all \( x_0 \in X \).

Definition 2. ([13]) Let \( (X,d) \) be a metric space. An operator \( A : X \to X \) is a weakly Picard operator (WPO) if the sequence \( \{ A^n(x_0) \}_{n \in N} \) converges for all \( x_0 \in X \) and the limit (which may depend on \( x_0 \)) is a fixed point of \( A \).

If \( A \) is a WPO, then we define the following operator \( A^\circ : X \to X \) by the relation:

\[ A^\circ (x) := \lim_{n \to \infty} A^n(x). \]
We observe that $A^\alpha(X) = F_A$. A generic example for a weakly Picard operator (WPO) is presented below.

**Example 1.** ([13]) Let $(X,d_i), i \in I$, be a family of metric spaces, $A_i : X_i \to X_i$ a family of the Picard operators and $x_i^*$ the unique fixed point of the operator $A_i$.

Let $X := \bigcup_{i \in I} \bigcup_{\lambda \in A_i} X_i^\lambda$, such that there exists a partition of $X$, defined by the relation:

\[
d(x,y) \geq \begin{cases} d_i(x,y) & \text{if } x,y \in X_i, \ i \in I \\ d_i(x,x_i^\lambda) + d_j(y,y_j^\alpha) + 1 & \text{if } i \neq j, \ x \in X_i, \ y \in X_j \end{cases}
\]

Then the operator $A$ is a WPO. Moreover, we have the following theorem of characterization of weakly Picard operators:

**Theorem 1.** (I.A. Rus [13]) Let $(X,d)$ a metric space and $A : X \to X$ an operator. The operator $A$ is WPO if and only if there exists a partition of $X$, $X = \bigcup_{\lambda \in A} X_{\lambda}$, such that

(a) $X_{\lambda} \in I(A)$, $\lambda \in A$;

(b) the restriction of $A$ to $X_{\lambda}$, $A_{|X_{\lambda}} : X_{\lambda} \to X_{\lambda}$ is PO for all $\lambda \in A$.

This theorem is very useful to prove that some operator are WPOs.

In what follows, we present the definition of a new class of operators, namely the $c$–weakly Picard operators ($c$–WPO) and we present two relevant examples (see [13]).

**Definition 3.** ([13]) Let $A : X \to X$ be a WPO and $c > 0$. The operator $A$ is $c$–WPO, if

\[
d(A^\alpha(x),A(x)) \leq c \ d(x,A(x)), \ \forall x \in X.
\]

**Definition 4.** ([13]) Let $A : X \to X$ be a PO and $c > 0$. The operator $A$ is $c$–PO, if

\[
d(A^\alpha(x)), A(x)) \leq c \ d(x,A(x)), \ \forall x \in X.
\]

**Example 2.** ([13]) Let $(X,d)$ a complete metric space and $A : X \to X$ an $\alpha$–contraction. Then $A$ is $c$–WPO with $c = (1–\alpha)^{-1}$.

**Example 3.** ([13]) Let $(X,d)$ be a complete metric space and $A : X \to X$ a closed operator. We suppose that there exists $\alpha \in [0,1)$ such that

\[
d(A^\alpha(x), A(x)) \leq \alpha \ d(x,A(x)), \ \forall x \in X.
\]

Then $A$ is $c$–WPO with $c = (1–\alpha)^{-1}$.

An important result contains the following theorem:

**Theorem 2.** (I.A. Rus [13]) Let $(X,d)$ be a metric space and $A_i : X \to X$, $i = 1,2$, two operators. If the following conditions are satisfied:

(i) the operators $A_i$ are $c_i$–WPOs, $i = 1,2$;

(ii) there exists $\eta > 0$ such that

\[
d(A_1(x), A_2(x)) \leq \eta, \ \forall x \in X
\]

then

\[
H(F_{A_1}, F_{A_2}) \leq \eta \cdot \max(c_1, c_2),
\]

where $H$ is the Pompeiu–Hausdorff metric.

**Proof.** Let $x \in F_{A_1}$, i.e. $x = A_1(x)$. Then we have

\[
d(x, A_1^\alpha(x)) \leq c_2 \ d(x, A_2(x)) = c_2 \ d(A_1(x), A_2(x)) \leq c_2 \eta.
\]

Also, let $y \in F_{A_2}$, i.e. $y = A_2(y)$ and in a similar way we obtain

\[
d(y, A_2^\alpha(y)) \leq c_1 \ d(y, A_1(y)) = c_1 \ d(A_2(y), A_1(y)) \leq c_1 \eta.
\]

Now, the proof of this theorem follows from the following lemma:

**Lemma 1.** (I.A. Rus [13]) Let $(X,d)$ be a metric space and $U$, $V \in P(X)$. If for a $\eta > 0$, the following conditions are satisfied:

(i) for all $u \in U$, there exists $v \in V$ such that $d(u,v) \leq \eta$;

(ii) for all $v \in V$, there exists $u \in U$ such that $d(u,v) \leq \eta$.

then

\[
H(U,V) \leq \eta.
\]

Now, from (2), (3) and the lemma 1 it results that

\[
H(F_{A_1}, F_{A_2}) \leq \eta \cdot \max(c_1, c_2),
\]

and the proof is complete. \(\square\)

The data dependence theorem from the next section is an application of this theorem.
\section{The main results}

We consider the functional-integral equation (1)

\[ x(t) = g(t, x(t), x(a)) + \int_{a}^{t} K(t, s, x(s), x(a))ds, \ t \in [a, b] \]

and applying the theorem 2 we obtain the conditions of data dependence of the solution of this equation.

We consider the Banach space \( X = (C[a, b], \| \cdot \|_{B}) \), where \( \| \cdot \|_{B} \) is the Bielecki norm, defined by the relation:

\[ \| x \|_{B} = \max_{\tau \in [a, b]} \left( x(\tau) e^{-\tau(a-a)} \right), \ \tau > 0. \] (4)

Relative to the integral equation (1), we suppose that

\begin{itemize}
  \item[(c1)] \( K \in C([a, b] \times [a, b] \times \mathbb{R}^{2}, \mathbb{R}) \), \( g \in C([a, b] \times \mathbb{R}^{2}, \mathbb{R}) \);
  \item[(c2)] there exists \( l_{g} > 0 \) such that \( \| g(t, u_{1}, v) - g(t, u_{2}, v) \| \leq l_{g} \| u_{1} - u_{2} \| \), for all \( t \in [a, b] \), \( u_{1}, u_{2}, v \in \mathbb{R} \);
  \item[(c3)] there exists \( l_{K} > 0 \) such that \( \| K(t, s, u_{1}, v) - K(t, s, u_{2}, v) \| \leq l_{K} \| u_{1} - u_{2} \| \), for all \( t, s \in [a, b] \), \( u_{1}, u_{2}, v \in \mathbb{R} \);
  \item[(c4)] there exists \( \tau > 0 \) such that \( l_{g} + \frac{l_{K}}{\tau} < 1 \).
\end{itemize}

For \( t = a \) we have \( x(a) = g(a, x(a), x(a)) \).

If we denote \( x(a) = \alpha, \alpha \in \mathbb{R} \), then we consider the following equation in \( \alpha \)

\[ \alpha = g(\alpha, \alpha, \alpha) \] (5)

and let \( S_{\alpha} \) be the solutions set of the equation (5).

If \( S_{\alpha} = \emptyset \), then the integral equation (1) has not solution.

We suppose that \( S_{\alpha} \neq \emptyset \) and we denote

\[ X_{\alpha} = \{ x \in (C[a, b], \| \cdot \|_{B}) / x(a) = \alpha \}. \]

Now, we have the following partition \( X = \bigcup_{\alpha \in \mathbb{R}} X_{\alpha} \).

Let \( A: C[a, b] \rightarrow C[a, b] \) be the operator defined by the relation:

\[ A(x)(t) = g(t, x(t), x(a)) + \int_{a}^{t} K(t, s, x(s), x(a))ds. \] (6)

The solutions set of the integral equation (1) coincides with the fixed points set of the operator \( A \), i.e. coincides with \( F_{\alpha} \).

Under the conditions (c1)–(c4) the operator \( A \) is a WPO and card \( F_{\alpha} = \text{card} \ S_{\alpha} \) (see [11]).

We estimate the following difference:

\[ |A(x)(t) - A(y)(t)| = \]

\[ = \left| g(t, x(t), x(a)) + \int_{a}^{t} K(t, s, x(s), x(a))ds - g(t, y(t), y(a)) + \int_{a}^{t} K(t, s, y(s), y(a))ds \right| \]

\[ \leq |g(t, x(t), x(a)) - g(t, y(t), y(a))| + \int_{a}^{t} |K(t, s, x(s), x(a)) - K(t, s, y(s), y(a))|ds \leq \]

\[ \leq l_{g} |x(t) - y(t)| + \int_{a}^{t} l_{K} |x(s) - y(s)|ds = \]

\[ = l_{g} |x(t) - y(t)| e^{-\tau(a-a)} e^{\tau(t-a)} + \]

\[ + \int_{a}^{t} l_{K} |x(s) - y(s)| e^{-\tau(s-a)} - e^{\tau(s-a)}ds \leq \]

\[ \leq l_{g} \| x - y \| \| e^{-\tau(a-a)} - e^{\tau(t-a)} \| \| e^{\tau(s-a)}ds = \]

\[ = l_{g} \| x - y \| e^{\tau(t-a)} + l_{K} \| x - y \| \frac{1}{\tau} e^{\tau(t-a)} = \]

\[ = \left( l_{g} + \frac{l_{K}}{\tau} \right) \| x - y \| e^{\tau(t-a)} \]

and we obtain

\[ |A(x)(t) - A(y)(t)| e^{-\tau(a-a)} \leq \left( l_{g} + \frac{l_{K}}{\tau} \right) \| x - y \|. \] (7)

Now using the Bielecki norm defined by the relation (4), in the left part of this inequality, it results that:

\[ |A(x) - A(y)| \leq \left( l_{g} + \frac{l_{K}}{\tau} \right) \| x - y \|. \] (8)

We observe that for a conveniently selected parameter \( \tau > 0 \), we have \( l_{g} + \frac{l_{K}}{\tau} < 1 \) and then the operator \( A \) is a contraction (the condition (c4)).

Now, we consider the following functional-integral equations

\[ x(t) = g_{i}(t, x(t), x(a)) + \int_{a}^{t} K_{i}(t, s, x(s), x(a))ds, \ t \in [a, b], \]

\( i \in \{1,2\} \) and we have the following theorem:

\textbf{Theorem 3.} We suppose that the conditions (c1)–(c4) are satisfied for the functions \( g_{i}, K_{i}, i=1,2 \). In addition we suppose that:

(i) there exists \( \eta_{1} > 0, \eta_{2} > 0 \), such that
Proceedings of the 13th WSEAS International Conference on COMPUTERS

$\|g_1(t,u,v) - g_2(t,u,v)\| \leq \eta_1, \forall t \in [a,b], \, u, v \in X$

and

$\|K_1(t,u,v) - K_2(t, u,v)\| \leq \eta_2, \forall t \in [a,b], \, u, v \in X$.

(i) \( S_{g_1} = S_{g_2} \),

then

$H_t(F_{A_1}, F_{A_2}) \leq \eta_1 + (b - a)\eta_2$ \frac{1}{1 - l_t}$,$

where

$l_t = \max\left\{ l_{g_1} + \frac{l_{K_1}}{\tau}, l_{g_2} + \frac{l_{K_2}}{\tau} \right\}$,

$\tau > 0$ suitable selected,

and \( H_t \) is the Pompeiu-Hausdorff metric corresponding to Bielecki norm \( \| \cdot \|_B \).

Proof. We consider the operators

$A_t \mid \bigcup_{a \in K} X_a : \bigcup_{a \in K} X_a \to \bigcup_{a \in K} X_a$,

defined by the relations:

$A_t(x)(i) := g_t(x(t), x(s)) + \int_{a}^{b} K_t(x(s), x(s))ds$

$t \in [a, b], \, i \in \{1, 2\}$.

(8)

Under the conditions (c1)–(c4) the operators

$A_t \mid \bigcup_{a \in K} X_a$ are WPOs and \( \text{card} F_{A_t} = \text{card} S_{g_t} \) (11).

Moreover, the operators $A_t \mid \bigcup_{a \in K} X_a$ are contractions with the constants

$l_{A_1} = l_{g_1} + \frac{l_{K_1}}{\tau}, \text{ respectively } l_{A_2} = l_{g_2} + \frac{l_{K_2}}{\tau}.$

Therefore these operators are $c_i$–weakly Picard operators, with $c_i = \frac{1}{1 - l_{A_i}}, \, i = 1, 2$.

Now, we apply the theorem 2 and we obtain

$H_t(F_{A_1}, F_{A_2}) \leq \frac{\eta_1 + (b - a)\eta_2}{1 - l_t}$, \text{i.e.}

$H_t(F_{A_1}, F_{A_2}) \leq \eta_1 + (b - a)\eta_2 \frac{1}{1 - l_t}$,

and the proof is complete. \( \square \)

References:


