Towards 2D electronic circuits in the spatial domain

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Abstract: Electronic circuits are, by nature, functions of one independent variable in the time domain. They operate (in the sense of differential operators) from an input signal $e(t)$ to an output signal $s(t)$. In this sense, we can speak of the electronic circuits of an ODE (ordinary differential equation) with its associated initial condition. We show that periodic networks of resistances (PNR) can be used to define a kind of 2D circuits in the spatial domain. These circuits operate as a PDE (partial differential equations) from an input signal $f(x, y)$ to an output signal $g(x, y)$. The two independent variables are now the coordinate axes $x$ and $y$. As to the initial conditions, they are replaced by boundary conditions. We present how to construct an electronic circuit of a linear PDE up to the fourth order. The discrete solution at each voltage node of the center of each cell composing this circuit converges towards the solution of the PDE $g(x, y)$, with its boundary conditions, for a given input signal $f(x, y)$ when the number of periodic cells increases.

Key–Words: Periodic networks of resistances, 2D circuits, Partial differential equations

1 Introduction

Electronic circuits are modelled by ODE (ordinary differential equations) of a time variable $t$ with its associated initial conditions [1]. We will show, in this paper, that Periodic Networks of Resistances (Called PNR) can be used to define a kind of 2-D networks in the spatial domain. These networks are then defined by PDE (partial differential equations) of two space variables $(x, y)$ with their associated boundary conditions. Table 1 summarizes the equivalence between regular networks and periodic networks of resistances.

<table>
<thead>
<tr>
<th>Networks</th>
<th>PNR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time domain</td>
<td>Spatial domain</td>
</tr>
<tr>
<td>1D Variable $t$</td>
<td>2D Variables $(x, y)$</td>
</tr>
<tr>
<td>ODE</td>
<td>PDE</td>
</tr>
<tr>
<td>Initial conditions</td>
<td>Boundary conditions</td>
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</tbody>
</table>

Table 1: Equivalence between networks and PNR.

Our method is based on the design of electronic circuits which “computed” locally the finite difference approximation of a partial differential equation. These circuits must have the properties of being possibly assembled then periodically recopied. The whole periodic network is then composed of the numerous repeated identical cells, except at the boundaries where the cell takes into account the boundary conditions. So we can say that PNR “solve” or “is described” by a PDE.

The 14+1 partial differential operators up to order 4 are summarized in table 2. It is sufficient to study 9 of them (at the left side of the table), all the other ones can be deduced by symmetry.

The following sections propose a study of eight unit circuits which “code” these 8 finite difference (FD) approximations. The name of the sections “Molecule $\alpha\beta^\gamma$” refer to the FD approximation of the derivative $\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha} \partial y^{\beta}}$ with an error proportional to the step size of order 2. The calculation of the harmonic

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</thead>
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<td>$\frac{\partial^2}{\partial x^2}$</td>
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<td>$\frac{\partial^6}{\partial x^6}$</td>
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Table 2: Partial differential operators up to order 4.
derivative (Molecule 20) is well known, all the other proposed circuits are originals. The study of each molecule is presented through an increasing difficulty and by gathering the similar solutions.

We introduce now some notations that we will use throughout the whole paper. In order to use the finite difference schemes, we divide the domain \((0, 1)^2\) into \(N^2\) interior points with spatial step size \(h = 1/(N + 1)\) in both \(x\)- and \(y\)-directions. The grid points \((x, y)\) are given by \(x_i = ih, i = 0, 1, 2, \ldots, N\) and \(y_j = jh, j = 0, 1, 2, \ldots, N\) where \(N\) is integer. Note that \(V_{i,j}\) is used to denote the finite difference approximation of \(v(ih, jh)\).

2 Molecule 20

The easiest computational molecule to implement under the form of an electronic circuit is the one associated with the second derivative. The molecule is symmetrical.

\[
\frac{\partial^2 v}{\partial x^2} \approx \frac{1}{h^2} \left\{ -2 \quad 1 \quad 0 \quad 1 \quad 0 \quad -2 \quad 1 \right\} \quad (1)
\]

We recall that a computational molecule is a graphical depiction of an approximate partial derivative formula. The previous one stands for:

\[
\frac{\partial^2 v}{\partial x^2} \approx \frac{1}{h^2} (v_{i-1,j} - 2v_{i,j} + v_{i+1,j})
\]

![Figure 1: Network of molecule 20.](image)

The elementary cell which “calculate” (1) is represented on Fig. 1 with its two adjacent cells. We call \(\rho_1\) the resistance along the paths located by the index 1 in the scheme and linking two adjacent potentials (i.e. \(\rho_1\) is the resistance between \(V_i\) and \(V_{i-1}\) or \(V_i\) and \(V_{i+1}\)). By applying the Kirchhoff Current Law (KCL) at node \(V_i\), rearranging some terms and dividing by \(h^2\), the equation of the cell \(i\) can be written under the form (2).

\[
\frac{1}{h^2} (V_{i-1} - 2v_i + V_{i+1}) = -\frac{1}{h^2} \rho_1 i_0 \quad (2)
\]

If we choose resistances which scaled down in \(h^2\) with the cell size, i.e. \(\rho_1 = -h^2 \rho_0\), the potential of a cell \(i\) estimates the (second) derivative of the function \(v(x)\) at a single point. The whole electronic circuit composed of \(N\) cells computes a finite difference approximation of the differential equation (3). The particular choice to take \(\rho_1 = -h^2 \rho_0\) instead of \(\rho_1 = h^2 \rho_0\) is very important, it allows to superpose the electrical network of this molecule to other one (all the circuit equations must have the same left hand side to be superposed).

\[
\frac{\partial^2 v}{\partial x^2} = \rho_0 i_0 \quad (3)
\]

The values of the resistances inside a cell depend only on the circuit topology and are easily expressed as a function of \(\rho_1\), here \(r_1 = r_2 = \rho_1/2\).

3 Molecule 40

The computational molecule of the fourth derivative is given by (4). The molecule is symmetrical and expands over two steps at each side.

\[
\frac{\partial^4 v}{\partial x^4} \approx \frac{1}{h^4} \left\{ -4 \quad 6 \quad -4 \quad 1 \right\} \quad (4)
\]

![Figure 2: Network of molecule 40.](image)

The elementary cell which “calculate” (4) is represented on Fig. 2 with its four adjacent cells. The resistances \(\rho_1\) connect the potentials \(V_i\) and \(V_{i±2}\) and the resistances \(\rho_2\) connect the potentials \(V_i\) and \(V_{i±1}\). Applying KCL at node \(V_i\) and doing some algebraic manipulations leads to (5).

\[
\frac{1}{h^4} \left[ 2V_i \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right) - \frac{1}{\rho_1} V_{i-2} - \frac{1}{\rho_2} V_{i-1} - \frac{1}{\rho_2} V_{i+1} - \frac{1}{\rho_1} V_{i+2} \right] = \frac{1}{h^4} i_0 \quad (5)
\]

If we choose \(\rho_1 = -h^4 \rho_0\) and \(\rho_2 = h^4 \rho_0/4\), then the potential at node \(V_i\) estimate the fourth derivative of the function \(v(x)\) at a given point (Eq. (6)).

\[
\frac{\partial^4 v}{\partial x^4} = \rho_0 i_0 \quad (6)
\]

The resistances of the cells can be taken as \(r_1 = r_3 = r_4 = r_6 = \rho_1/4\) and \(r_2 = r_5 = \rho_2/2\).
4 Molecule 10

The molecule of the first derivative is represented by (7). This molecule is not symmetrical and thus cannot be represented by a symmetrical network.

\[
\frac{\partial v}{\partial x} \approx \frac{1}{2h} \left\{ -1 -0 -1 \right\} \tag{7}
\]

To implement the molecule, we must construct an asymmetrical circuit. The proposed one uses the potential \(-V_{i,j}\). It is obtained by a linear VCVS \(e_0\) of value \(-1\). (VCVS stands for Voltage-Controlled Voltage Source). The circuit is drawn on Fig. 3.

![Network of molecule 10](image)

Figure 3: Network of molecule 10.

The resistances \(\rho_1\) connect the potentials \(V_i\) and \(V_{i+1}\) and the resistance \(\rho_2\) connect the potentials \(V_i\) and \(-V_{i+1}\). Applying KCL at node \(V_i\) and after some algebraic manipulations, one can obtain (8).

\[
\frac{1}{2h} \left[ V_i \left( \frac{1}{r_0} + \frac{2}{\rho_1} + \frac{1}{\rho_2} \right) - V_{i-1} \left( \frac{1}{\rho_1} \right) + V_{i+1} \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \right] = \frac{1}{2h} e_0 \tag{8}
\]

If we choose \(\rho_1 = 2h\rho_0\), \(\rho_2 = 2h\rho_0/2\), and \(r_0 = -2h\rho_0/4\), then the potential at node \(V_i\) estimates the first derivative of the function \(v(x)\) at a given point (Eq. (9)).

\[
\frac{\partial v}{\partial x} = \rho_0 i_0 \tag{9}
\]

The resistances of the cells can be taken as \(r_1 = r_3 = \rho_1/2\) and \(r_2 = r_4 = \rho_2/2\).

5 Molecule 30

The molecule of the third derivative is represent by (10). Like every odd order derivative, the third molecule is not symmetrical and thus, it cannot be represented by a symmetrical network. Moreover, it expands over two steps at each side.

\[
\frac{\partial^3 v}{\partial x^3} \approx \frac{1}{2h^3} \left\{ -1 -2 -0 -2 -1 \right\} \tag{10}
\]

The proposed circuit is represented on Fig. 4. The circuit is a combination of the method used in molecule 40 to obtain the value of the potential, and those used in molecule 10 to get an asymmetrical value in the molecule. The paths 1 and 2 code the outer values of the molecule \(\{ -1,1 \}\), while the paths 3 and 4 code its inner values \(\{ 2, -2 \}\). Note that the minus sign is inverted between these outer and inner values, this is translated into a direction change in the network (compare paths 2 and 4).

![Network of molecule 30](image)

Figure 4: Network of molecule 30.

The resistances \(\rho_1\) connect the potentials \(V_i\) and \(V_{i+2}\) and the resistance \(\rho_2\) connect the potentials \(V_i\) and \(-V_{i+2}\). In a similar way, the resistances \(\rho_3\) connect the potentials \(V_i\) and \(V_{i+2}\) and the resistance \(\rho_2\) connect the potentials \(V_i\) and \(-V_{i+1}\). Applying KCL at node \(V_i\) and after some algebraic manipulations, one can obtains (11).

\[
\frac{1}{2h^3} \left[ V_i \left( \frac{1}{r_0} + \frac{2}{\rho_1} + \frac{1}{\rho_2} + \frac{2}{\rho_3} + \frac{1}{\rho_4} \right) - V_{i-2} \left( \frac{1}{\rho_1} \right) + V_{i+2} \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) + V_{i-1} \left( \frac{1}{\rho_4} - \frac{1}{\rho_3} \right) - V_{i+1} \left( \frac{1}{\rho_3} \right) \right] = \frac{1}{2h^3} i_0 \tag{11}
\]

If we choose \(\rho_1 = 2h^3\rho_0\), \(\rho_2 = 2h^3\rho_0/2\), \(\rho_3 = 2h^3\rho_0/2\), \(\rho_4 = 2h^3\rho_0/4\) and \(r_0 = -2h^3\rho_0/12\), then the potential at node \(V_i\) estimates the third derivative of the function \(v(x)\) at a given point (Eq. (12)).

\[
\frac{\partial^3 v}{\partial x^3} = \rho_0 i_0 \tag{12}
\]

The resistances of the cells can be taken as \(r_{\{1,3,7,9\}} = \rho_1/4\), \(r_{\{4,6,10,12\}} = \rho_2/4\), \(r_{\{2,8\}} = \rho_3/2\) and \(r_{\{5,11\}} = \rho_4/2\).

6 Molecule 11

The molecule 11 derivative is represented by (13). This molecule is the tensor product of two asymmetrical molecules (the molecules 10 and 01) and is there-
fore symmetrical.

\[
\frac{\partial^2 V}{\partial x \partial y} \approx \frac{1}{4h^2} \begin{bmatrix}
-2 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & -3
\end{bmatrix}
\] (13)

The proposed circuit is given on Fig 5. Following the same convention and notation as before, \(\rho_1\) correspond to the resistance values between the potentials \(V_{i,j}\) and \(V_{i\pm1,j,\pm1}\), and \(\rho_2\) are the resistance values between the potentials \(V_{i,j}\) and \(V_{i\pm1,j\pm1}\). The electronic network which evaluates the molecule 11 is drawn on the center cell of Fig. 5, its eight adjacent cells are also represented. To facilitate the understanding and the analysis of the network, only the center cell is completely drawn. On all the other cells, only the resistances along the paths which link there cell center \((V_{i-1,j+1}, V_{i,j+1}, \text{etc})\) to \(V_{i,j}\) have been represented.

![Figure 5: Network of molecule 11.](image)

Applying KCL at node \(V_i\) and after some algebraic manipulations, one can obtain (14).

\[
\frac{1}{4h^2} \left[ V_{i,j} \left( \frac{2}{\rho_1} + \frac{2}{\rho_2} \right) - \frac{1}{\rho_1} V_{i+1,j+1} - \frac{1}{\rho_1} V_{i-1,j+1} - \frac{1}{\rho_1} V_{i,j-1} - \frac{1}{\rho_1} V_{i,j-1} - \frac{1}{\rho_1} V_{i+1,j-1} \right] = \frac{1}{4h^2} i_0
\] (14)

If we choose \(\rho_1 = -4h^2 \rho_0\) and \(\rho_2 = 4h^2 \rho_0\), the potential at node \(V_i\) estimates the 1–1 partial derivative of the function \(v(x)\) at a given point (Eq. (15)).

\[
\frac{\partial^2 v}{\partial x \partial y} = \rho_0 i_0
\] (15)

The actual resistances are represented by thick vertical and horizontal segments. The resistances represented by long segments can be taken as \(r_{(\text{long})} = \rho_1/2\), while the ones represented by short segments can be taken as \(r_{(\text{short})} = \rho_2/2\). For example, along the path from \(V_{i,j}\) to \(V_{i+1,j+1}\), there are 4 resistances (of the same value) in series which are in parallel with 4 others, so the equivalent resistance is well \((4 \times \rho_1/2)/(4 \times \rho_1/2) = \rho_1\).

![Figure 6: Path linking \(V_{i,j}\) and \(V_{i+1,j+1}\).](image)

The network has been designed in such a way that one gets a maximum number of resistances having the same value, up to the sign. It seems that such resistances are easier to integrate in practical circuits. It is not difficult to derive an equivalent network, from the proposed one, with a much weaker number of resistances.

### 7 Molecule 22

The molecule 22 derivative is represented by (16). This molecule is the tensor product of two symmetrical molecules (the molecules 20 and 02) and is therefore symmetrical.

\[
\frac{\partial^4 v}{\partial x^2 \partial y^2} \approx \frac{1}{h^4} \begin{bmatrix}
-2 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & -2
\end{bmatrix}
\] (16)

The proposed circuit is given on Fig 7. As shown in the network, \(\rho_1\) correspond to the resistance values between the potentials \(V_{i,j}\) and \(V_{i\pm1,j}\) and between \(V_{i,j}\) and \(V_{i,j\pm1}\), and \(\rho_2\) are the resistance values between the potentials \(V_{i,j}\) and \(V_{i\pm1,j\pm1}\) and between \(V_{i,j}\) and \(V_{i\pm1,j\pm1}\). The electronic network which evaluate the molecule 22 is drawn on the center cell of Fig. 7, its eight adjacent cells are also represented. To facilitate the understanding and the analysis of the network, only the center cell is completely drawn.
metrical molecule and one asymmetrical molecule (the molecules 20 and 01) and is therefore asymmetrical.

\[
\frac{\partial^3 v}{\partial x^2 \partial y} \approx \frac{1}{2h^3} \begin{pmatrix}
1 & -2 & 3 \\
0 & 0 & 0 \\
-1 & 2 & -3
\end{pmatrix}
\]  

(19)

To define its electronic circuit, the molecule is splitted into 3 smaller molecules crossing the point \((i, j)\), the vertical one \((-2, 0, 2)\), and the molecules along the two diagonals \((-1, 0, 1)\). Each submolecule can then be implemented by the same method as the one used for the molecule 10.

A careful drawing of the location of resistances to obtain a periodicity of the cells lead to the network of Fig. 8. Only the center cell is completely drawn to make easier the reading of the network. The resistance between one of the 9 paths linking the potentials of adjacent cell to \(V_{i,j}\) are called \(\rho_\alpha\). The bold index \(\alpha\) on the scheme indicates their corresponding path.

Figure 8: Network of molecule 21.

Applying KCL at node \(V_i\) and after some algebraic manipulations, one can obtain (17).

\[
\frac{1}{h^3} \left[ V_{i,j} \left( \frac{4}{\rho_1} + \frac{4}{\rho_2} \right) - \frac{1}{\rho_1} V_{i+1,j} - \frac{1}{\rho_1} V_{i,j+1} \right] = \frac{1}{3} \rho_0 i_0
\]

If we choose \(\rho_1 = h^4 \rho_0 / 2\) and \(\rho_2 = -h^4 \rho_0\), the potential at node \(V_i\) estimates the 2–2 partial derivative of the function \(v(x)\) at a given point (Eq. (18)).

\[
\frac{\partial^4 v}{\partial x^2 \partial y^2} = \rho_0 i_0
\]

(18)

The actual resistances are represented by thick vertical and horizontal segments. The eight resistances represented by very long segments can be taken as \(r_{\{\text{very long}\}} = \rho_1 / 2\), while all the other ones (i.e. resistances represented by long and short segments) can be taken as \(r_{\{\text{long, short}\}} = \rho_2 / 2\). The explanation of this choice is identical to that of the molecule 11.

8 Molecule 21

The molecule 21 derivative is represented by (19). This molecule is the tensor product of one sym-
braic manipulations, one can obtain (20).

\[
\frac{1}{2h^3} \left[ V_{i,j} \left( \frac{1}{r_0} + \frac{2}{\rho_1} + \frac{1}{\rho_2} \right) + \frac{2}{\rho_3} + \frac{1}{\rho_4} + \frac{2}{\rho_5} + \frac{1}{\rho_6} \right] + V_{i,j-1} \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) - V_{i,j+1} \left( \frac{1}{\rho_1} \right) + V_{i-1,j+1} \left( \frac{1}{\rho_4} - \frac{1}{\rho_3} \right) - V_{i+1,j-1} \left( \frac{1}{\rho_3} \right) + V_{i+1,j+1} \left( \frac{1}{\rho_6} - \frac{1}{\rho_5} \right) - V_{i-1,j-1} \left( \frac{1}{\rho_5} \right) \right] = \frac{1}{2h^3} i_0 \tag{20}
\]

If we choose \( r_0 = -2h^3/12 \) and the sets of \( \rho_0 \) so that,

\[
\begin{align*}
\rho_1 &= 2h^3 \rho_0/2 \\
\rho_2 &= 2h^3 \rho_0/4 \\
\rho_3 &= 2h^3 \rho_0 \\
\rho_4 &= 2h^3 \rho_0/2 \\
\rho_5 &= 2h^3 \rho_0 \\
\rho_6 &= 2h^3 \rho_0/2
\end{align*}
\]

the potential at node \( V_i \) estimates the 2–1 partial derivative of the function \( v(x) \) at a given point (Eq. (21)).

\[
\frac{\partial^4 v}{\partial x^3 \partial y} = \rho_0 i_0 \tag{21}
\]

The actual resistances are represented by segments between two dots. The 6 resistances along the pathes 1 and 2 can be taken as \( r_{(1), (2)} = \rho_1/2 \), while the resistances along the pathes 3, 4, 5 and 6 can be taken as \( r_{(1), (2), (3), (4), (5), (6)} = \rho_2/3 \). The value of the actual resistances along the pathes define, by periodicity, the resistances in the elementary cell.

## 9 Molecule 31

The molecule 21 derivative is represented by (22). This molecule is the tensor product of two asymmetrical molecule (the molecules 30 and 01) and is therefore symmetrical.

\[
\frac{\partial^4 v}{\partial x^3 \partial y} \approx \frac{1}{4h^4} \left[ \begin{array}{cccccc}
& 2 & & 0 & & 2 \\
& 0 & & 0 & & 0 \\
& 1 & & 1 & & 2 \\
& 2 & & 0 & & 0 \\
& 0 & & 0 & & 0 \\
& 1 & & 1 & & 0
\end{array} \right] \tag{22}
\]

The molecule is separated into two submolecules around the center point \((i, j)\). The first submolecule is the inner one \((-2, 2, -2, 2)\), it can be defined under a circuit form from the circuit of the molecule 11 by inverting and scaling the resistance \( \rho_0 \). The circuit of the second molecule, the outer one \((1, -1, 1, -1)\) is drawn on Fig 9. As indicated in the network, \( \rho_1 \) correspond to the resistances between the potentials \( V_{i,j} \) and \( V_{i \pm 2, j \mp 1} \) and \( \rho_2 \) are the resistances between the potentials \( V_{i \mp 2, j \pm 1} \).

![Figure 9: Network of outer submolecule of molecule 31.](image)

The circuit of the molecule 31 is the “schematic sum” of the network of Fig. 9 and the network of molecule 11 (with resistances renamed \( \rho'_1 \) and \( \rho'_2 \)). A “schematic sum” means that the two circuits must be superposed with separated dots in the cell border, and that these two circuits have the same current source \( i_0 \). Applying KCL at node \( V_i \) and after some algebraic manipulations, one can obtain (23).

\[
\frac{1}{4h^4} \left[ V_{i,j} \left( \frac{2}{\rho_1} + \frac{2}{\rho_2} \right) + V_{i,j} \left( \frac{2}{\rho_1} + \frac{2}{\rho_2} \right) \right. \\
&\left. - \frac{1}{\rho_1} V_{i+2,j+1} - \frac{1}{\rho_1} V_{i-2,j+1} \\
&\left. - \frac{1}{\rho_1} V_{i-2,j-1} - \frac{1}{\rho_2} V_{i+2,j-1} \\
&\left. - \frac{1}{\rho_1} V_{i+1,j+1} - \frac{1}{\rho_2} V_{i-1,j+1} \\
&\left. - \frac{1}{\rho_2} V_{i+1,j-1} - \frac{1}{\rho_2} V_{i+1,j-1} \right] = \frac{1}{4h^4} i_0 \tag{23}
\]

If we choose \( \rho_1 = -4h^3 \rho_0, \rho_2 = 4h^3 \rho_0 \) and the corresponding resistances \( \rho'_1 \) and \( \rho'_2 \) of the molecule 11 so that \( \rho'_1 = 4h^3 \rho_0/2 \) and \( \rho'_2 = -4h^3 \rho_0/2 \), the potential at node \( V_i \) estimates the 3–1 partial derivative of the function \( v(x) \) at a given point (Eq. (24)).

\[
\frac{\partial^4 v}{\partial x^3 \partial y} = \rho_0 i_0 \tag{24}
\]

The actual resistances of Fig. 9 are represented by segments between two dots. The 4 resistances along the pathes 1 and 2 can be taken as \( r_{(1), (2)} = \rho_1/4 \). The value of the actual resistances along the pathes define, by periodicity, the resistances in the elementary cell.
10 Simulation and comments

We have developed a program which generate the Spice network corresponding to the chosen finite difference discretisation scheme from a (linear) partial differential equation. The generated file contained the $N$ or $N^2$ cells of the domain, and the cells at the boundaries to take account of the boundary conditions. All the circuits defined in the previous sections have been independently checked by simulating the PDE:

$$\frac{\partial^{\alpha+\beta} V}{\partial x^\alpha \partial y^\beta} = \alpha!\beta!,$$

with $\alpha + \beta \leq 4$ on the domain $\Omega = [0, 1] \times [0, 1]$, with the right boundary conditions. The Spice solution for the potential $V_{i,j}$ has been compared with the PDE solutions $v(x, y) = u^\alpha y^\beta$ for different mesh sizes. On these academic examples, there is no significant difference between the solutions.

The case $\alpha = \beta = 0$ corresponds to the Molecule 00, which is the trivial molecule $(1)$. The unit circuit which “codes” the equation $V_{i} = \rho_{0}i_{0}$ is given in Fig 10. This case is important, it allows to manage the circuit which “codes” the equation and facilitates the management of the difference between the solutions.

![Figure 10: Network of molecule 00.](image)

This paper considers the problem of designing the electronic linear circuit of a (linear) PDE. Reference [2] considers the reverse problem, i.e. to build the (linear) PDE of an electronic linear circuit (The method is based on an extension of an homogenization modelling theory initially developed to study composite materials).

Unfortunately, the two theories operate on circuits with very different topologies. Thus, the common networks they can treat simply reduce to one circuit. Indeed, The periodicity of the electronic network implies that each node $n$ located on the boundary of the unit cell has its counterpart $n'$ on the opposite side. The method used in [2] imposes that each such node couple $(n, n')$ is linked by at least a crossing path. All the networks developed here lead to unit cells which have not generally this property, the exception is the circuit of molecule 20 (strictly, the molecule 40 can be analysed too by taking 2 adjacent cells).

11 Conclusion

We have proposed a concept of periodic network of resistances (PNR) to define a kind of 2D electronic circuit in the spatial domain. We adopt an electronic point of view, i.e. designing the electronic circuits and simulating them. The matrix equations of our periodic circuit are the same that the corresponding set of finite difference equation to resolve a given PDE. However, the accuracy and the stability of the numerical schemes have to be carefully studied.

By doing a “schematic sum” of some elementary networks defined in the previous sections, it is possible to build the 2D electronics circuit which is a solution of a given linear partial differential equation, for example the plate equation:

$$\frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = \frac{P}{D}.$$  

Moreover, generalization to non constant coefficient PDE or PDE with an arbitrary function in the right hand side is faisible by the same method. By using multiplior, even nonlinear PDE could be manageable. All these generalizations suppose implicitly that the finite difference equations relative to the periodic network are mathematically valid.

An example of application of PNR could be the real time control of a system, with a PDE control law implemented under the form of an analog integrated circuit instead of a numerical one. It is difficult to quantify the gain in speed, that may be of a factor 1000.

Of course, a practical implementation of the method described in this paper under the form of integrated circuits is challenging, some of the numerous problems are the integration of the many NIC (Negative Impedance Convertor) to simulated the negative resistances: How to choose the magnitude of the elements ? Should the scaling operate on the resistances or on the current sources ? What is the effect of the scattering of the component values (some finite difference schemes are very sensitive to it, but finite difference schemes deriving from physical equations have “generally” a good behavior).

References:
