A discrete IS-LM model with tax revenues

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Abstract: In this paper, we study a discrete IS-LM model with tax revenues. Considering its parameters as variables, we analyze the existence of the Neimark-Sacker bifurcation. We find the normal form and its Lyapunov coefficient. Finally, using a program in Maple 11 we make some numerical simulations that verify the theoretical results.

Key-Words: IS-LM model, tax revenues, Neimark-Sacker bifurcation.

1 Introduction

Recently, there has been great interest in dynamical characteristics of economic models with time delay and time delays have been incorporated into economic models by many authors. In [2], the authors consider an IS-LM model with delayed taxation revenues, which is augmented by a government budget constraint in the tradition of the well known Schinasi paper [10]. Varying the length of lag and applying the “stability switch criteria”, they proved that the equilibrium may lose or gain its local stability and that the existence of limit cycles generated by subcritical and supercritical Hopf bifurcation is obtained. In [7] an IS-LM model with the same lag in the tax revenues and the capital accumulation equation is presented. The authors analyzed the quantitative behavior of the model via the Hopf bifurcation of stability switch criteria. In [4] an IS-LM model with distributed tax collection lag is considered offering an explanation of the multiperiodicity and irregularity in business cycles.

Almost all real economic processes have the state variables defined at different moments, thus the discrete models are important in obtaining the practical results. In the present paper, we will use the discretization method from [3].

After this introduction, in Section 2 we study the discrete IS-LM model with tax revenues. This model is obtained by the discretization of the continuous IS-LM model with tax revenues. The functions that describe the model are the investment function I and the liquidity function L. Also, some numerical simulations are performed using a program in Maple 11. Concluding comments are presented in Section 3.

We start from the following dynamical system [2]:

\[
\begin{align*}
\dot{Y}(t) &= a[I(Y(t), R(t)) - S(Y^D(t)) - T(t) + g] \\
\dot{R}(t) &= b[L(Y(t), R(t)) - M(t)] \\
\dot{M}(t) &= c(g - T(t)),
\end{align*}
\]

(1)

with: the national income Y(t), the tax revenues T(t) given by:

\[ T(t) = (1 - \varepsilon)qY(t) + \varepsilon q \int_0^t k(s)Y(t - s)ds, \]

the disposable income given by:

\[ Y^D(t) = Y(t) - T(t), \]

the tax rate q, the saving function S(Y^D(t)), the interest rate R, the real money supply M (prices are fixed at unity).

Further on, we assume that investment I is a function of national income Y and the interest rate R, i.e. I(Y, R), the liquidity preference function L is a function of national income Y and the interest rate, i.e. L(Y, R), g represents the government expenditure, a, b, c represent respectively the speed of adjustment in the goods market, depreciation rate of the interest rate, the speed of adjustment in the money market.
The following is assumed about the derivatives:

\[ I_Y = \frac{\partial I(Y, R)}{\partial Y} > 0, \quad I_R = \frac{\partial I(Y, R)}{\partial R} > 0, \]
\[ L_Y = \frac{\partial L(Y, R)}{\partial Y} > 0, \quad L_R = \frac{\partial L(Y, R)}{\partial R} > 0, \]
\[ S_Y = \frac{\partial S}{\partial Y} = s, 0 < s < 1, q \in (0, 1). \]

Let \( k : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \), be the density of distribution that verifies the following properties:

\[ k(s) \geq 0, s \in \mathbb{R}_+, \int_0^\infty k(s)ds = 1, \int_0^\infty sk(s)ds < \infty \]

and it is called kernel.

If \( k \) is the Dirac density of distribution, then:

\[ \int_0^t k(s)Y(t - s)ds = Y(t - \tau) \]

where \( \tau \geq 0 \) is the delay.

If \( k \) is the exponential density of distribution, then:

\[ k(s) = \alpha e^{-\alpha s}, \alpha > 0 \]

and the variable \( z(t) = \int_0^\infty k(s)Y(t - s)ds \) satisfies the differential equation:

\[ \dot{z}(t) = \alpha (Y(t) - z(t)). \]

If \( k \) is the Erlang density of distribution:

\[ k(s) = \alpha^2 s e^{-\alpha s}, \alpha > 0 \]

then \( z(t) = \int_0^\infty k(s)Y(t - s)ds \) satisfies the following relations:

\[ \dot{z}(t) = \alpha (u(t) - z(t)) \]
\[ \dot{u}(t) = \alpha (Y(t) - u(t)) \]
\[ u(t) = \alpha \int_0^\infty e^{-\alpha s}Y(t - s)ds \]

With these assumptions dynamic system (1) is given by:

\[ \dot{Y}(t) = a[I(Y(t), R(t)) - s(1-q)Y(t) + qY(t)] + g \]
\[ \dot{R}(t) = b(L(Y(t), R(t)) - M(t)) \]
\[ \dot{M}(t) = c(g - T(t)), \]

the dynamic model with continuous time and delay;

\[ \dot{Y}(t) = a[I(Y(t), R(t)) - (s+1-\varepsilon)(1-s)qY(t) - (1-s)\varepsilon qY(t-\tau) + g] \]
\[ \dot{R}(t) = b(L(Y(t), R(t)) - M(t)) \]
\[ \dot{M}(t) = c(g - (1-\varepsilon)qY(t) - \varepsilon qY(t-\tau)), \]

the dynamic model with continuous time and exponential density of distribution;

\[ \dot{Y}(t) = a[I(Y(t), R(t)) - (s+1-\varepsilon)(1-s)qY(t) - (1-s)\varepsilon qz(t) + g] \]
\[ \dot{R}(t) = b(L(Y(t), R(t)) - M(t)) \]
\[ \dot{M}(t) = c(g - (1-\varepsilon)qY(t) - \varepsilon qz(t)) \]
\[ \dot{z}(t) = \alpha((1-\varepsilon)qY(t) - (1-\varepsilon)z(t)) \]

the dynamic model with continuous time and Erlang density of distribution.

The model (3) with the functions:

\[ I(Y, R) = a_1 Y^{a_2} R^{a_3}, L(Y, R) = b_1 Y + b_2 Y^{b_3}(R-b_4)^{-b_5} \]

was analyzed in [8].

The discrete models are obtained by the discretization of the models (2), (3), (4), (5) with the adjustment coefficient different from the ones above. For simplicity, we will employ the same coefficients:

\[ Y(n+1) = Y(n) + a[I(Y(n), R(n)) - s(1-q)Y(n) + qY(n)] + g \]
\[ R(n+1) = R(n) + b(L(Y(n), R(n)) - M(n)) \]
\[ M(n+1) = M(n) + c(g - qY(n)), n \in N \]

the discrete IS-LM model;
\[ Y(n + 1) = Y(n) + a[I(Y(n), R(n)) - (s + 1 - \varepsilon) Y(n)] \\
\times (1 - s)q Y(n) - (1 - s)\varepsilon q Z(n) + g] \\
R(n + 1) = R(n) + b(L(Y(n), R(n)) - M(n)) \\
M(n + 1) = M(n) + c(g - \varepsilon q Y(n) - \varepsilon Z(n)) \\
z(n + 1) = z(n) + \alpha(1 - \varepsilon) q Y(n) - (1 - \varepsilon) z(n) \\
u(n + 1) = u(n) + \alpha((1 - \varepsilon)q Y(n) + \varepsilon z(n) - u(n))
\]

(7)

the discrete IS-LM model with delay corresponding to (4):

\[ Y(n + 1) = Y(n) + a[I(Y(n), R(n)) - (s + 1 - \varepsilon) Y(n)] \\
\times (1 - s)q Y(n) - (1 - s)\varepsilon q Z(n) + g] \\
R(n + 1) = R(n) + b(L(Y(n), R(n)) - M(n)) \\
M(n + 1) = M(n) + c(g - \varepsilon q Y(n) - \varepsilon Z(n)) \\
z(n + 1) = z(n) + \alpha(1 - \varepsilon) q Y(n) - (1 - \varepsilon) z(n) \\
u(n + 1) = u(n) + \alpha((1 - \varepsilon)q Y(n) + \varepsilon z(n) - u(n))
\]

(8)

the discrete IS-LM model with delay corresponding to (5):

2 The analysis of the discrete IS-LM model with tax revenues (6)

We analyze system (6). Because \( I_R < 0 \) we assume that:

\[ \lim_{R \to 0^+} I(\frac{q}{q}, R) \geq \frac{1 - q}{q} g, \]

\[ \lim_{R \to -\infty} I(\frac{q}{q}, R) \geq \frac{1 - q}{q} g. \]

Under the previous conditions, the model (6) has the equilibrium state \((Y_0, R_0, M_0)\), so that: \( Y_0 = \frac{q}{q}, \)

\( R_0 \) is the solution of \( I(\frac{q}{q}, R) = \frac{1 - q}{q} g \) and \( M_0 = L(Y_0, R_0) \).

The linearization of system (6) in the neighborhood of the equilibrium yields:

\[ v_1(n + 1) = a_{11}v_1(n) + a_{12}v_2(n) + a_{13}v_3(n) \]

\[ v_2(n + 1) = a_{21}v_1(n) + a_{22}v_2(n) + a_{23}v_3(n) \]

\[ v_3(n + 1) = a_{31}v_1(n) + a_{32}v_2(n) + a_{33}v_3(n) \]

(9)

where

\( a_{11} = 1 + a(I_1 - s(1 - q) - q), a_{12} = aI_2, a_{13} = 0, \)

\( a_{21} = bL_1, a_{22} = 1 + bL_2, a_{23} = -b, \)

\( a_{31} = -eq, a_{32} = 0, a_{33} = 1 \)

(10)

and

\[ I_1 = I_Y(Y_0, R_0), I_2 = I_R(Y_0, R_0), I_3 = L_Y(Y_0, R_0), I_4 = L_R(Y_0, R_0). \]

The characteristic equation of system (9) is given by:

\[ \lambda^3 - A_1\lambda^2 + A_2\lambda - A_3 = 0 \]

(11)

where

\[ A_1 = 1 + a_{11} + a_{22}, \]

\[ A_2 = a_{11}a_{22} - a_{12}a_{21} + a_{11} + a_{22}, \]

\[ A_3 = a_{11}a_{22} - a_{12}a_{21} + a_{12}a_{23}a_{31}; \]

From (10) and (12), we get:

\[ A_1 = 3 + (I_1 - s(1 - q) - q)a + L_2b, \]

\[ A_2 = 3 + 2(I_1 - s(1 - q) - q)a + 2L_2b - \text{abs}(1 - q)L_2, \]

\[ A_3 = 1 + (I_1 - s(1 - q) - q)a - L_2b - \text{abs}(1 - q)L_2 + + abcqI_2. \]

(13)

According to the Schur criterion [6], equation (11) has the roots in modulus less than 1, if and only if:

\[ |A_3| < 1, |A_1 - A_3| < 2, 1 - A_2 + A_3(A_1 - A_3) > 0. \]

Then, if the parameters of the model satisfy the relations (14), the equilibrium point \((Y_0, R_0, M_0)\) is asymptotically stable.

We analyze the roots of equation (11) considering the parameters a and b as fixed and the parameter c as variable. We denote by:

\[ d_1 = a_{11}a_{22} - a_{12}a_{21}, d_2 = qa_{12}a_{23}. \]

(15)

From (13) and (15), we have:

\[ A_3 = d_1 - cd_2. \]

The following proposition holds:
Proposition 1 (i) If the parameters $a$, $b$, $c$ satisfy the relations:

$$|d_1 - cd_2| < 0, |A_1 - d_1 + cd_2| < 2,$$  

then equation (11) has one root in modulus less than 1 and two complex roots in modulus equal to 1.

(ii) If $c_0$ is one solution of (17) then for $c = c_0 - \alpha$ equation (11) has one root in modulus less than 1 and the complex conjugate roots are in modulus equal to 1.

(iii) The value $c_0$ is a Neimark-Sacker bifurcation for equation (11).

The proof of the proposition results from (11) and from the definition of the Neimark-Sacker bifurcation [6].

In what follows, using the method from [6], [8], we find the normal form of system (6) for the Neimark-Sacker bifurcation given by $c_0$.

We consider $A$ the matrix of the linear system (9) with the coefficients (10) and $c = c_0 + \alpha$, where $c_0$ satisfies (17) and $|\alpha|$ is sufficiently small.

We have:

Proposition 2 (i) The eigenvector corresponding to the eigenvalue $\mu$ is the nontrivial solution of the system $A \tilde{I} = \mu \tilde{I}$ and has the components:

$$l_1 = -a_{12}(\mu - a_{33}), l_2 = -(\mu - a_{11})(\mu - a_{33}),$$

$$l_3 = -a_{12}a_{33};$$  

(ii) The eigenvector corresponding to the eigenvalue $\bar{\mu}$ is the nontrivial solution of the system $A^T \tilde{m} = \bar{\mu} \tilde{m}$ and has the components:

$$m_1 = \frac{\bar{\mu} - a_{22}}{a_{12}V} l_1 + \frac{a_{23}}{a_{12}} l_2 + \frac{a_{23}}{a_{33}} \bar{l}_3,$$

$$m_2 = \frac{1}{V}, m_3 = \frac{a_{23}}{(\bar{\mu} - a_{33})V},$$

where $V = \frac{\bar{\mu} - a_{22}}{a_{12}} l_1 + \frac{a_{23}}{a_{12}} l_2 + \frac{a_{23}}{a_{33}} l_3.$

Moreover, the relation $l_1 m_1 + l_2 m_2 + l_3 m_3 = 1$ holds.

We denote by:

$$L_{20} = L_{YY}(Y_0, R_0), L_{11} = L_{YR}(Y_0, R_0),$$
$$L_{02} = L_{RR}(Y_0, R_0),$$
$$L_{30} = L_{YY}(Y_0, R_0), L_{21} = L_{YYR}(Y_0, R_0),$$
$$L_{12} = L_{YRR}(Y_0, R_0), L_{03} = L_{RRR}(Y_0, R_0),$$

and

$$B_1 = \begin{pmatrix} I_{20} & I_{11} \\ I_{11} & I_{02} \end{pmatrix}, B_2 = \begin{pmatrix} L_{20} & L_{11} \\ L_{11} & L_{02} \end{pmatrix},$$
$$C_1 = \begin{pmatrix} I_{30} & I_{21} \\ I_{21} & I_{12} \end{pmatrix}, D_1 = \begin{pmatrix} I_{12} & I_{03} \\ I_{12} & I_{03} \end{pmatrix},$$
$$C_2 = \begin{pmatrix} L_{30} & L_{21} \\ L_{21} & L_{12} \end{pmatrix}, D_2 = \begin{pmatrix} L_{12} & L_{12} \\ L_{12} & L_{03} \end{pmatrix}. $$

We consider $l = (l_1, l_2)^T$, $m = (m_1, m_2)^T$, where $l_1, l_2, m_1, m_2$ are given by (18) and (19) and

$$B^i(l, \bar{l}) = (l^T B l)^i, B^i (\bar{l}, \bar{l}) = (B^T B l)^i, i = 1, 2,$$
$$C^i(l, \bar{l}) = (l^T C l + l_2 D l)^i, i = 1, 2,$$
$$g_{20} = (B^1(l, \bar{l}), B^2(l, \bar{l}))m, g_{11} = (B^1(\bar{l}, \bar{l}), B^2(\bar{l}, \bar{l}))m, g_{02} = (B^1(\bar{l}, \bar{l}), B^2(\bar{l}, \bar{l}))m, $$

(23)
We denote by:

\[
A(\mu^2) = \begin{pmatrix}
\mu^2 - a_{11} & -a_{12} & 0 \\
-a_{21} & \mu^2 - a_{22} & -a_{23} \\
-a_{31} & 0 & \mu^2 - a_{33}
\end{pmatrix},
\]

\[
A(1) = \begin{pmatrix}
1 - a_{11} & -a_{12} & 0 \\
-a_{21} & 1 - a_{22} & -a_{23} \\
-a_{31} & 0 & 1 - a_{33}
\end{pmatrix},
\]

\[
A(\bar{\mu}^2) = \begin{pmatrix}
\bar{\mu}^2 - a_{11} & -a_{12} & 0 \\
-a_{21} & \bar{\mu}^2 - a_{22} & -a_{23} \\
-a_{31} & 0 & \bar{\mu}^2 - a_{33}
\end{pmatrix}
\]

and

\[
w_{20} = A(\mu^2)^{-1}h_{20}, w_{11} = A(1)^{-1}h_{11},
\]

\[
w_{02} = A(\bar{\mu}^2)^{-1}h_{02},
\]  
(24)

where

\[
h_{20} = (h_{120}, h_{220}, h_{320})^T, \]

\[
h_{11} = (h_{111}, h_{211}, h_{311})^T, h_{02} = (h_{102}, h_{202}, h_{302})^T
\]

and

\[
g_{21} = (B^1(I, w_{20}), B^2(I, w_{20}))m + 2(B^1(I, w_{11}), B^2(I, w_{11}))m + (C^1(I, l, \bar{I}), C^2(I, l, \bar{I}))l.
\]  
(25)

Using the method from [6], [8], for the determination of the normal form, we obtain:

**Proposition 3** (i) The normal form of system (6) is:

\[
z(n + 1) = \mu z(n) + \frac{1}{2}g_{20}z(n)^2 + g_{11}z(n)\tilde{z}(n) + \frac{1}{2}g_{02}\tilde{z}(n)^2 + \frac{1}{2}g_{21}z(n)^2\tilde{z}(n),
\]

(26)

where \(z(n) \in C\) and the coefficients are given by (23) and (25);

(ii) System (6) in the neighborhood of the state equilibrium \((Y_0, R_0, M_0)\) is:

\[
Y(n) = Y_0 + l_1z(n) + \tilde{l}_1\tilde{z}(n) + \frac{1}{2}w_{120}z(n)^2 + w_{111}z(n)\tilde{z}(n) + \frac{1}{2}w_{102}\tilde{z}(n)^2 + w_{111}z(n)\tilde{z}(n) + \frac{1}{2}w_{102}z(n)^2
\]

\[
R(n) = R_0 + l_2z(n) + \tilde{l}_2\tilde{z}(n) + \frac{1}{2}w_{220}z(n)^2 + w_{211}z(n)\tilde{z}(n) + \frac{1}{2}w_{202}\tilde{z}(n)^2
\]

\[
M(n) = M_0 + l_3z(n) + \tilde{l}_3\tilde{z}(n) + \frac{1}{2}w_{320}z(n)^2 + w_{311}z(n)\tilde{z}(n) + \frac{1}{2}w_{302}\tilde{z}(n)^2
\]

where \(z(n)\) is one solution of (26) and the coefficients are given by (24);

(iii) The Lyapunov coefficient associated to the normal form (26) is given by:

\[
C_1(\alpha) = \frac{g_{20}(\alpha)g_{11}(\alpha)}{2(\mu(\alpha) - \bar{\mu}(\alpha))(\bar{\mu}(\alpha) - 1)} + \frac{|g_{11}(\alpha)|^2}{(1 - \bar{\mu}(\alpha))} + \frac{|g_{02}(\alpha)|^2}{2(\mu(\alpha) - \bar{\mu}(\alpha))} + \frac{g_{21}(\alpha)}{2};
\]

(iv) If \(\theta_0 = arg(\mu(0))\), \(L_0 = Re(e^{-i\theta}C_1(0))\) and \(L_0 < 0(> 0)\) in the neighborhood of the equilibrium state \((Y_0, R_0, M_0)\), then there is a stable (unstable) limit cycle.

The numerical simulation was made using a program in Maple 11. For:

\[
I = a_1Y^{a_2}R^{-a_3}, L = b_1Y + b_2(r - b_3)^{-b_4}
\]

where \(a_1 = 0.38, a_2 = 1.05, a_3 = 0.83, b_1 = 0.07, b_2 = 1, b_3 = 0.003, b_4 = -1.2, s = 0.5, q = 0.18, g = 10, a = 0.96, b = 0.8,\) we obtain the following results: \(Y_0 = 55.5, R_0 = 2.33, M_0 = 6.64, c_0 = 0.3560.\) The Lyapunov coefficient is \(L_0 = 0.337\) and in the neighborhood of the equilibrium state there is an unstable limit cycle.

For \(c = c_0 + \beta, \beta = 0.001,\) the following trajectories are displayed: \((n, Y(n)), (n, R(n)), (n, M(n))\) in Fig. 1, Fig. 2, Fig. 3. In the figures Fig. 4, Fig. 5, Fig. 6 are displayed: \((Y(n - 1), Y(n)), (R(n - 1), R(n)), (M(n - 1), M(n)).\)
For \( c = c_0 - \beta, \beta = 0.001 \) in Fig 7, Fig 8, Fig 9 the following trajectories: \((n, Y(n)), (n, R(n)), (n, M(n))\) are displayed. In the figures Fig 10, Fig 11, Fig 12 are displayed: \((Y(n - 1), Y(n)), (R(n - 1), R(n)), (M(n - 1), M(n))\).

These graphics justify the behavior of the model's solutions as obtained in the theoretical section.

3 Conclusion

The discrete time IS-LM model with tax revenues is a complex model with many parameters. The model allows us to study the real process using the temporal numeric series of the income, the interest rate, the money stock, the liquidity.

The analysis of the model leads to different scenarios by considering the adjustment coefficient of the equation which describes the dynamics of the money stock as variable.

In the present paper we have analyzed the discrete model with the parameter \( c \) as variable and we have shown the existence of the Neimark-Sacker bifurcation. The normal form of the model has also been presented. We will carry out the same analysis for the discrete model with delay. In a future paper the scenario for which the model has a chaotic behavior will be analyzed.

Acknowledgements: The research was supported by the Grant with the title "The qualitative analysis and numerical simulation for some economic models which contain evasion and corruption", The National University Research Council from Ministry of Education and Research of Romania (grant No. 1085/2008).

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