Minimum risk portfolios using MMAR

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Abstract: In traditional portfolio optimization the aim is to construct a portfolio of assets which simultaneously optimize a blend of high return and small risk. Within the classical Markowitz model, the efficient frontier identifies the set of portfolios that a rational investor considers according to his degree of risk aversion. The basic assumption of the standard theory is that asset returns are multivariate normal, what is shown to be inconsistent with empirical evidence by a growing number of studies, which devote a particular emphasis on the role played by the variance as a measure of risk. In order to improve the portfolio selection, we suggest to model the asset price dynamics by a multifractal stochastic process. The empirical analysis concerned a selection of major assets of the U.S. market and the European market. In both cases the proposed methodology shows considerable improvements of performance on the holding periods of one, two, three, six and twelve months.

Key–Words: Multifractal Model of Asset returns, Portfolio’s selection, Hurst’s exponent, risk measure, Sharpe ratio

1 Introduction

Optimal asset allocation deals with how to divide the investor’s wealth across some asset-classes in order to maximize the investor’s gain. One criterion is to maximize the utility of the portfolio of assets at the end of the investment period. The classical mean-variance model, which involves maximizing the portfolio return and minimizing the risk, was proposed by Markowitz [13] in 1952.

Many different approaches to the problem have been proposed (see [14] [16] [11] [15] [7] [9] for examples), trough model with increasing mathematical sophistication. In the most basic setting the planning horizon is just a single period, and transaction costs are ignored. This allows some of the basic ideas to be discussed but limits the realism of the models. In order to improve the portfolio selection, we suggest to model the asset price dynamics by a multifractal stochastic process, i.e. a model which composes a fractional Brownian motion [12] (fBm) of parameter $H$ with a multifractal trading time. According to this novel approach, the exponent $H$ of the fBm, replaces the traditional standard deviation. Notice this replacement is not intended in the sense of a risk measure change; in fact $H$ is not positive homogeneous\(^1\). What we suggest here is a radical change of point of view: the risk is conceived as no long associated to dispersion around a mean value but to the smoothness of the process and it increases with the jaggedness . In section 2 we briefly recall the mean-variance approach. In section 3 we introduce the multifractal formalism and in the next one we apply the methodology to the portfolio analysis. Some numerical results are given to show the proposed methodology. Section 5 summarizes our results and compares the models through a back-test. Section 6 concludes.

2 Mean Variance Models

Let us consider a set of the available investment universe comprised of $N$ risky assets. A portfolio can be represented as a $N$-vector $w = (w_1, w_2, ..., w_N)$ where each weight $i$ represents the percentage of the $i$-th assets in the portfolio subject to the condition

$$\sum_{i=1}^{N} w_i = 1$$  \hspace{1cm} (1)

\(^1\)A risk measure $\rho$ satisfies the following properties:

1. Subadditivity. $\rho(X + Y) \leq \rho(X) + \rho(Y)$

2. Positive homogeneity. For any positive real number $c$, $\rho(cX) = c\rho(X)$

where $X$ and $Y$ are random variables.
Suppose that the assets included in the portfolio have expected returns \( \mu = (\mu_1, \mu_2, ..., \mu_N)^t \) and the covariance matrix given by

\[
\Sigma = \begin{bmatrix}
\sigma_{11} & \cdots & \sigma_{1N} \\
\vdots & \ddots & \vdots \\
\sigma_{N1} & \cdots & \sigma_{NN}
\end{bmatrix}
\]

\( \sigma_{ij} \) being the covariance between asset \( i \) and \( j \). Obviously

\[
\sigma_{ii} = \sigma_i^2
\]

and

\[
\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j
\]

where \( \rho_{ij} \) is the correlation between the asset \( i \) and \( j \). It follows that the return of the portfolio is the random variable \( R_{\Pi} = w' \tilde{R} \); the portfolio’s expected return and variance are given by

\[
\mu_{\Pi} = w' \mu
\]

\[
\sigma_{\Pi}^2 = w' \Sigma w
\]

Equipped with the above notation, in Markowitz’s framework the goal is to solve the constrained minimization problem:

\[
\min_w w' \Sigma w (2)
\]

subject to

\[
\mu_0 = w' \mu
\]

\[
w' 1 = 1, \quad 1' = [1, 1, \cdots, 1]
\]

where \( \mu_0 \) is the target main return.

This is the classical version of the mean-variance optimization problem, usually denoted as risk minimization formulation. Using the Lagrange multipliers the solution of the quadratic optimization reads as

\[
w = g + h \mu_0
\]

where \( g \) and \( h \) are the two vectors

\[
g = \frac{1}{ac-b^2} \cdot \Sigma^{-1} [c1 - b \mu]
\]

\[
h = \frac{1}{ac-b^2} \cdot \Sigma^{-1} [a \mu - b1]
\]

Each feasible portfolio can be represented as a point in the Cartesian plane, with abscissa given by the expected return and ordinate equal to the standard deviation. So, the so called efficient frontier is obtained by solving the optimization problem for different choices of \( \mu_0 \).

Other constraints that are usually used in combination with the mean-variance optimization problem are:

- \( w \geq 0 \) when short-selling is not allowed;
- \( L_i \leq w_i \leq U_i \) when maximal holdings on single asset must be controlled. \( L_i \) is the lower bound of the holdings of asset \( i \) and \( U_i \), similarly, the upper bound;
- \( L_i \leq \sum_{i\in I} w_i \leq U_i \) to constrain the exposure to a specific set \( I \) of the available investment universe \( I \). \( L_i \) and \( U_i \) denotes the maximum exposure to \( I_i \).

### 3 Multifractal model

A very interesting property of financial time series is the long term dependence (LD) of absolute or squared returns, or global dependence, documented by an increasing number of contributions published in many high quality finance journals in the last ten years. A model which seems to be promising in this regard is the Multifractal Model of Asset Returns (MMAR), introduced by Calvet et al. [3] [4] [5] and Calvet and Fisher [6] as a development of Mandelbrot’s findings in the field of multifractal measures. It is a continuous-time process that captures the thick tails and the volatility persistence exhibited by many financial time series. The model is constructed by compounding a Fractional Brownian Motion with a random time-deformation stochastic process, which is specified to be multifractal. Under some additional conditions the model can be proved to be arbitrage free and, thus, consistent with the martingale theory. The introduction of the multifractal time deformation, which has the relevant property of preserving a form of time invariance called multiscaling, succeeds in combining extreme returns, long-memory and clustering in volatility and produces moments scaling as a power law of the time horizon. Beside the MMAR, only GARCH and FIGARCH exhibit the volatility clustering in return series, while having the martingale property. Elliot and Van der Hoecck [8] propose an option pricing formula based on multifractality, Jamdee and Los [10] simulate their monofractal European options pricing formula and Elliot and Chan extend and incorporate the mono-fractality into the American options pricing formula.

Denoted by \( \{ P(t); 0 \leq t \leq T \} \) the price of the financial asset quoted with respect to the clock time in the interval \([0, T]\), the MMAR considers the stochastic process

\[
X(t) = \ln P(t) - \ln P(0)
\]

and assumes \( \{ X(t) \} \) to be multifractal. In a general way, a multifractal process is characterized by the following three definitions:
In this special case, comparing (5) and (6) we obtain

\[ \tau(q) = Hq - 1 \]

Hence, the scaling function is linear when the process is unifractal, while the introduction of a time deformation implies the nonlinearity of the scaling function. The most relevant properties of the scaling function are the following:

\[ \tau\left(\frac{1}{H}\right) = 0 \]  \hspace{1cm} (7)

\[ \tau_X(q) = \tau_H(Hq) \]  \hspace{1cm} (8)

Equation (7) allows us to estimate the Hurst exponent of the fBm of the MMAR, and the equation (8) shows the relationship between the scaling function of time series price and the scaling function of the trading time. Using properties (7) and (8), we can estimate the parameters (\(H\) and \(\tau(t)\)) and synthesize the MMAR. If we want to perform an analysis of the geometric properties of sample paths in the MMAR we must study the multifractal spectrum of the process, which is defined by the following theorem:

**Theorem 5.** The multifractal spectrum \(f(\alpha)\) is the Legendre transform of the scaling function \(\tau(q)\)

\[ f(\alpha) = \inf_{\alpha} [q(\alpha) - \tau(q)] \]  \hspace{1cm} (9)

While a unifractal process presents a linear scaling function and a multifractal spectrum folding on a single point \(H : f(\alpha) = H\), a multifractal process displays a nonlinear scaling function and a concave spectrum \(^2\).

### 3.1 Multifractal Formalism

The scaling functions notion is extracted from multifractal formalism.

**Definition 4.** A stochastic process \(\{X(t)\}\) is called multifractal if it has stationary increments and satisfies

\[ E(|X(0)|^q) = c(q)t^{\tau(q) + 1} \]  \hspace{1cm} (5)

where, \(E(\cdot)\) is the expectation operator and \(c(q)\) is called the prefactor. \(B\) and \(Q\) are intervals on the real line. Moreover, \(B\) and \(Q\) have positive lengths, and \(0 \in B, [0, 1] \in Q\).

\(\tau(q)\) is called the scaling function and takes into account the influence of the trading time \(t\) on moments \(q\). In the case of fBm it is not difficult to show that the scaling function is linear. In fact, a fBm with Hurst exponent \(H\) satisfies

\[ X(t) = t^H X(1) \]  \hspace{1cm} (6)

which implies that

\[ E(|X(t)|^q) = t^{Hq} E(|X(1)|^q) \]

In this special case, comparing (5) and (6) we obtain the form of the prefactor \(c(q)\) and of the scaling function \(\tau(q)\)

\[ c(q) = E(|X(1)|^q) \]

\[ \tau(q) = Hq - 1 \]

Really, Bianchi and Pianese provide evidence that also non multifractal processes show non linear scaling function and concave spectra \([1]\).
4 Portfolio analysis under multifractality

Let the portfolio’s value at time $t$ be

$$
\Pi(t) = \sum_{i=1}^{N} w(i) X_i(t)
$$

(11)

where $N$ stands for the number of assets, $w_i$ is the fraction of wealth allocated on asset $i$ (subject to $\sum_{i=1}^{N} w(i) = 1$) and $X_i(t)$ denotes the price series at time $t$.

Under the assumption of a multifractal $\Pi(t)$, the partition function (10) becomes:

$$
\Pi S_q(T, \Delta t) = \sum_{i=0}^{K-1} |w_1 dX_1^i + \ldots + w_N dX_N^i|^{q+1}
$$

where

$$
dX_j^i = X_j((i+1)\Delta t) - X_j(i\Delta t) \quad \text{for} \quad j = 1, \ldots, N
$$

Linearizing the previous relation

$$
\log(E[\Pi S_q(T, \Delta t)]) = \tau(q) \log \Delta t + \log[T \cdot \tau(q)]
$$

by (7) it is easy to obtain the estimated portfolio’s $H$. According to this novel approach, since the log price is modeled by compounding a fractional Brownian motion with the multifractal measure, it seems natural to view the parameter $H$ of the fBm as a sort of risk indicator. In fact, $H$ rules out the smoothness of the sample paths of the process: low values ($H < 0.5$) indicate high volatile markets, whereas high values ($H > 0.5$) indicate low volatile phases characterized by persistence in trends. The case $H = 0.5$ reduces to the Brownian motion in multifractal time. Another way to justify the interpretation of $H$ as an indicator of risk resides in the well-known relationship

$$
\sigma_H^2 = \Gamma(1 - 2H) \frac{\cos(\pi H)}{\pi H}
$$

linking $H$ itself and the standard deviation in the case of fractional Brownian motion in physical time.

With the above considerations, in this framework, the goal is to solve the constrained minimization problem:

$$
\max_{\Pi} w H_{\Pi}
$$

(12)

subject to

$$
\mu_{\Pi} = w' \mu
$$

$$
w' 1 = 1, \quad 1' = [1, 1, \ldots, 1]
$$

where $\mu_{\Pi}$ is the target main return. Notice that, $H$ being insensitive to the direction of the trend, the solution of the optimization problem (12) by itself does not imply an upward movement, but adding the constraint $\mu_{\Pi} > 0$ ensures to filter only the positive cases. In our application the following additional constraints, already discussed in Section 2, are also considered:

$$
L_i \leq w_i \leq U_i
$$

The optimization problem can be improved on replacing the objective function (12) with the following multi-objective function:

$$
\left\{ \begin{array}{l}
\max_{\Pi} w H_{\Pi} \\
\min_{\Pi} C_{\Pi}
\end{array} \right.
$$

(13)

where $C_{\Pi}$ denotes the portfolio’s covariance matrix.

5 Data and empirical results

In order to evaluate the performance of the "multifractal" optimal portfolios solving the problems (12) and (13) with respect to the classical mean-variance optimal portfolio (2), a large set of actual financial data were used to calculating the relative return function over a time horizon of one year.

The empirical analysis was developed along the following lines:

1) random selection of two subsets of forty-five assets listed in the U.S. Nasdaq $^3$ and in the European Footsie 100 and DAX $^4$. The indices were chosen on the basis of the availability of large sample sets of historical prices.

2) Solution of the optimization problems (2), (12) and (13), subject to the following constraints:

$$
\begin{array}{l}
\mu_{\Pi} = \mu_0 \\
\sum_{i=1}^{N} w_i = 1 \\
0 \leq w_i \leq 0.4 \\
0.6 \leq \sum_{j \in U_1} w_j \leq 0.75 \\
0.25 \leq \sum_{j \in U_2} w_j \leq 0.4
\end{array}
$$

$^3$30 from Nasdaq Ind., denoted by $U_1$ and 15 from Nasdaq Telecommunications denoted by $U_2$.

$^4$30 from Footsie, denoted by $E_1$ and 15 from DAX, denoted by $E_2$. 
for the USA case;
\[
\begin{align*}
\mu_\Pi = \mu_0 & \quad \mu_0 \in \{0.0012, 0.0038, 0.0063\} \\
\sum_{i=1}^{45} w_i = 1 & \\
0 \leq w_i \leq 0.4 & \\
0.6 \leq \sum_{j \in E_1} w_j \leq 0.75 & \\
0.25 \leq \sum_{j \in E_2} w_j \leq 0.4 & 
\end{align*}
\]

for the European case.

For each case, the thresholds of return were fixed as the minimal, the mean and the maximal obtainable return along the efficient frontier \(^5\).

3) To compare the behavior of the model in bear and bull market two different investment holding periods were considered: 01/01/2007 - 12/31/2007 for the bull market and 01/01/2008 - 12/31/2008 for the bear market. The portfolios were selected (starting day) on December 31, 2006 and 2007, respectively. \( H \) was estimated from a sample of 750 observations preceding the starting day; as usual, a sample of 120 observations was considered for the estimation of the standard deviation.

4) The performances of the three portfolios were evaluated by comparing their Sharpe ratios. This choice deserves a clarification: although in our approach \( H \) is representative of the risk level, we use the standard deviation (denominator of the Sharpe ratio) with the sole intent to make results comparable. In other words, the standard deviation is employed in a conventional way to provide a measure of performance through a widely accepted indicator. Another way of comparing the results would require one to estimate \( H \) from the Markowitz portfolio and to implement it in an indicator of performance, but the cost would be loosing the immediacy of reading.

Table 1 displays the Jaccard indices \(^6\) for each of the three couples obtained by the portfolios.

The generally low values of the Jaccard indices indicate the effectiveness of the new procedures in selecting portfolios which are very different from the portfolio obtained using the classical Markowitz technique. Notice that even when the Jaccard coefficient equals one, the portfolios can differ because of the weights resulting by the optimization.

Figures 1 and 2 display the returns with respect to the starting day obtained by the three portfolios over one year. A pattern which seems to hold for almost all the paths is that the returns of the Markowitz portfolio remain below those of the \( H \)-based portfolios in the initial period (till about 150 trading day). This effect could be explained by the change in the level of memory, which is likely to vary over time \(^2\) more sensitively than the standard deviation.

Figures 3 - 8 synthesize the Sharpe ratios. In almost all the cases the results seem in favour of the \( H \)-based portfolios. In detail, in the bear market (European case) the Markowitz portfolio shows a higher negative Sharpe ratio and the performance improves only for an holding period of one year and only with

\(^5\)Really, the extreme values of the thresholds differ from the actual minimal and the maximal values of about 10% in order to ensure the convergence of the optimization algorithms.

\(^6\)The Jaccard index relative to the sets \( A \) and \( B \) is defined as \( \frac{A \cap B}{A \cup B} \) and is used to measure the similarity of the two sets.

<table>
<thead>
<tr>
<th>( \mu_0 = 0.0012 )</th>
<th>Markowitz max H</th>
<th>max H min C</th>
<th>max H min C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Markowitz max H</td>
<td>1</td>
<td>0.2667</td>
<td>0.6471</td>
</tr>
<tr>
<td>Markowitz max H</td>
<td>1</td>
<td>0.3158</td>
<td></td>
</tr>
<tr>
<td>Markowitz max H</td>
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<table>
<thead>
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<th>( \mu_0 = 0.0016 )</th>
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<th>max H min C</th>
<th>max H min C</th>
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</thead>
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<td>0.3333</td>
<td>0.8182</td>
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<tr>
<td>Markowitz max H</td>
<td>1</td>
<td>0.3846</td>
<td></td>
</tr>
<tr>
<td>Markowitz max H</td>
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<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \mu_0 = 0.0020 )</th>
<th>Markowitz max H</th>
<th>max H min C</th>
<th>max H min C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Markowitz max H</td>
<td>1</td>
<td>0.5000</td>
<td>0.7778</td>
</tr>
<tr>
<td>Markowitz max H</td>
<td>1</td>
<td>0.5556</td>
<td></td>
</tr>
<tr>
<td>Markowitz max H</td>
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<td></td>
</tr>
</tbody>
</table>

Table 1: Europe Jaccard Index

<table>
<thead>
<tr>
<th>( \mu_0 = 0.0012 )</th>
<th>Markowitz max H</th>
<th>max H min C</th>
<th>max H min C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Markowitz max H</td>
<td>1</td>
<td>0.3871</td>
<td>0.6389</td>
</tr>
<tr>
<td>Markowitz max H</td>
<td>1</td>
<td>0.4286</td>
<td></td>
</tr>
<tr>
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<table>
<thead>
<tr>
<th>( \mu_0 = 0.0038 )</th>
<th>Markowitz max H</th>
<th>max H min C</th>
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</tr>
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<tbody>
<tr>
<td>Markowitz max H</td>
<td>1</td>
<td>0.3043</td>
<td>0.7143</td>
</tr>
<tr>
<td>Markowitz max H</td>
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<td>0.3846</td>
<td></td>
</tr>
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<table>
<thead>
<tr>
<th>( \mu_0 = 0.0063 )</th>
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<th>max H min C</th>
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<td>0.6000</td>
<td>1.0000</td>
</tr>
<tr>
<td>Markowitz max H</td>
<td>1</td>
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<td></td>
</tr>
<tr>
<td>Markowitz max H</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: USA Jaccard Index

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\(^7\)Remind that the multifractal model is written as a fractional Brownian motion in multifractal time. The parameter \( H \) of the fBm rules out the dependence (or memory) of the increment process.
respect to the portfolio (13). In all the cases the portfolio (12) displays the best performance. For the bull market (USA case) the situation does not change too much: generally the best performance corresponds to the portfolio (13), for all the considered thresholds of return.

6 Conclusion

In this paper we have developed two techniques to address the problem of portfolio selection under the assumption of a multifractal market. Using the parameter $H$ in place of the standard deviation and optimizing with respect to it, we improved the performances of the classical mean-variance based selection. Our work is a first attempt to justify from an empirical viewpoint the use of parameter $H$ for evaluating the risk. Obviously, as previously noted, even if we cannot speak of $H$ in terms of risk measure, the main purpose of this work is to solicit a debate about the suitability of the definition of risk measure with the suggestions came from the observed financial data. The results here discussed surely deserve further developments within a theoretical framework.

7 Bibliography

References:


Figure 1: Europe Relative Returns

Figure 2: USA Relative Returns