Data Reduction for Signals Observed in Colored Noise

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Abstract: This article treats data reduction in array processing for the spatially colored noise case. The purpose is to reduce the computational complexity of the applied signal processing algorithms by mapping the data into a space of lower dimension by means of a linear transformation. We discuss ways to implement the transformation and show that it suffices to estimate the array covariance matrix instead of the noise covariance matrix in the design process of the optimal transformation. Computer simulations are given that illustrate the problem of interference from out-of-band-sources that result when a beamspace transformation is designed to focus on a particular sector.

Key-Words: algorithms by mapping, covariance matrix, beamspace transformation.

1 Introduction
The computational complexity of the algorithms applied to the array signal processing problem is heavily dependent on the number of sensors in the array. Since large arrays with many sensors are preferable from an estimation accuracy viewpoint, accuracy and computational complexity are conflicting issues. This has led to different schemes for dimension reduction via linear transforms, beamspace transformations, which reduce the computational load. Besides a reduction in computational complexity, a beamspace transformation can have other advantageous effects, such as reduced bias, reduced sensitivity to directional interference, etc. [4, 7]. Different criteria can be used when deriving the transform. If one knows from which angular sectors that signals may emanate from, a possible approach is to design a transformation that focuses on these sectors. This can, e.g., be accomplished by a bank of conventional beamformers, the output of which are collected in a new vector, with reduced size. A more sophisticated variation of this theme is to design a beamspace transformation that maximizes the signal-to-noise ratio for signals impinging from the preselected sectors [12], or minimizes the interference power under the constraint that signals from the sectors are left un-distorted by the transform. The latter method is easily implemented by a bank of linearly constrained minimum variance beamformers, see, e.g., [10, 11]. Yet another method is to design a transformation that preserves the Cramér-Rao bounds (CRB), for the parameters of interest, which is the approach taken in the present paper. This criterion is also considered in [1, 2, 8] for the white-noise case, i.e., an equal amount of uncorrelated noise at each sensor, and a stochastic signal model. This paper concentrates on the spatially colored noise case, which is more realistic in any scenario, due to, e.g., directional interference from other sources (‘out-of-band sources’), mutual coupling between sensors, etc. The derived transform depends, in addition to the unknown DOAs, on the color of the noise process. However, it is shown herein that it is possible to implement the proposed transform without knowledge of the noise color and the exact directions of arrival; a fact that has great practical implications in a real scenario. The paper presents a design approach and computer simulations that support the theoretical results. The simulations evaluate the approach against the method of spheroidal sequences [4], and the method presented in [1, 8] with respect to the effect of out-of-band sources. The outline of the paper is as follows. Section 2 discusses the signal model, formulates the problem and derives the main results of the paper.
Section 3 treats practical implementation issues and Section 4 presents the computer simulations. Finally, conclusions are given in Section 5.

2 Model optimal

Consider an array of \( m \) sensors receiving \( p \) planar, narrowband, waveforms from the directions \( \{ \theta_1, \cdots, \theta_p \} \). The sensor outputs are modeled by the relation

\[
\tilde{y}(t) = \tilde{A}(\theta_0) s(t) + \tilde{e}(t),
\]

where \( \tilde{A}(\theta_0) \) is the array steering matrix,

\[
\tilde{A}(\theta_0) = \begin{bmatrix} \tilde{a}(\theta_1) \cdots \tilde{a}(\theta_p) \end{bmatrix},
\]

and the source signals are collected in the vector

\[
s(t) = \begin{bmatrix} s_1(t) \cdots s_p(t) \end{bmatrix}^T.
\]

The parameter vector \( \theta_0 \) is a column vector containing the true directions of arrival. The notation \( \tilde{x} \) is used to designate that the quantity, containing the true directions of arrival. The beamspace transformation by

\[
\begin{align*}
C & = \text{beamspace transformation} \\
& \text{of sensors} \to \text{complex vector space,}
\end{align*}
\]

is applied. The beamspace transformation matrix satisfies the following condition:

\[
\tilde{R} = \tilde{A}(\theta_0) R_{sa} \tilde{A}^H(\theta_0) + \tilde{Q} \tag{4}
\]

In general, to avoid ambiguous parameter estimates, some sort of structure must be imposed on the noise covariance matrix, see, e.g., [8,5]. However, since the main question here is preservation of the CRB of the transformed data, we do not treat the problem of parameter identifiability herein. Now, we introduce a linear transformation of the data from the complex \( m \)-dimensional vector space, \( \mathbb{C}^m \) (\( m \) being the number of sensors) to the complex \( n \)-dimensional vector space \( \mathbb{C}^n \) where \( n \leq m \). Denoting the \( m \times n \) beamspace transformation by \( T \), the beamspace signal model becomes

\[
y(t) = T^* \tilde{A}(\theta_0) s(t) + T^* \tilde{e}(t) = \tilde{A}(\theta_0) s(t) + e(t),
\]

where \( A = T^* \tilde{A} \) is the beamspace steering matrix and \( e(t) \) is the resulting beamspace noise. Based on \( N \) snapshots of the beamspace data, \( y(t) \), the directions of arrival, \( \theta_0 = [\theta_0_1, \cdots, \theta_0_p]^T \) are to be estimated.

Before stating and proving the main result of the paper, we briefly discuss the white-noise case, which is addressed in [1]. In this case, the spatial noise covariance matrix is a scaled version of the identity matrix, and the array covariance matrix is given by

\[
\tilde{R} = \tilde{A}(\theta_0) R_{sa} \tilde{A}^* (\theta_0) + \sigma^2 \tilde{I} \tag{6}
\]

where \( \sigma^2 \) is the variance of the additive thermal noise always present in the receiving equipment in a sensor array. Note also that the white-noise model assumes an equal amount of white-noise at each sensor. In order to retain the white-noise model in beamspace, an orthogonality constraint must be imposed on the beamspace transformation matrix: \( T^* T = I \). In [1], it is shown that the optimal beamspace transformation matrix satisfies the condition

\[
\forall (\lambda) \ni R \left[ \tilde{a}(\theta_1) \cdots \tilde{a}(\theta_p) \right] \geq \left[ \tilde{a}(\theta_1) \cdots \tilde{a}(\theta_p) \right] \tag{7}
\]

In the above equation, \( \forall (\lambda) \) denotes the range space of \( X \) while \( \tilde{A}(\theta) \) is the derivative of the steering vector with respect to \( \theta \). Thus, to obtain a beamspace CRB which equals the elementspace CRB, the range space of the beamspace transformation matrix should include the subspace spanned by the steering vectors and the derivative of the steering vectors evaluated at the true directions of arrival. It should be noted that the minimal beamspace dimension that satisfies the condition in Eq. (7) is \( n = 2p \).

We now return to the colored noise case. The following theorem states a condition on the range space of the beamspace transformation that is sufficient to guarantee that the beamspace CRB equals the elementspace CRB.

Theorem 1. The CRB for the estimate of \( \theta_0 \) in beamspace is equal to the CRB in element space, provided that the beam space transformation matrix \( T \) satisfies the following condition:

\[
\forall (\lambda) \ni R \left( \tilde{Q}^{-1} U(\theta_0) \right) = R \left( \tilde{R}^{-1} U(\theta_0) \right),
\]

where \( U(\theta_0) \) is defined as

\[
U(\theta_0) = \left[ \tilde{a}(\theta_1) \cdots \tilde{a}(\theta_p) \tilde{a}(\theta_1) \cdots \tilde{a}(\theta_p) \right]
\]

Proof. Without loss of generality, partition the beamspace transformation matrix in the following way:

\[
T = \begin{bmatrix} \tilde{Q}^{-1} UB : C \end{bmatrix} = [T_1 : T_2],
\]

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where \( B \), with dimension \( 2p \times 2p \), is a full-rank matrix and \( C \), with dimension \( m \times (n-2p) \), is an arbitrary matrix. For convenience, the dependence on \( \theta_0 \) is suppressed. The partitioning reflects the condition \( \mathbb{R}(T) \supseteq \mathbb{R}(\widetilde{Q}^{-1}U) \). The beamspace signal becomes

\[
y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} T_1^* \tilde{y}(t) \\ T_2^* \tilde{y}(t) \end{bmatrix}.
\]  

(9)

Concentrating on \( T_1^* \tilde{y}(t) \), we can write

\[
T_1^* \tilde{y}(t) = T_1^* \tilde{y}_w(t),
\]  

(10)

where

\[
T_w = \widetilde{Q}^{-1/2}UB, \quad \tilde{y}_w = \widetilde{Q}^{-1/2} \tilde{y}(t).
\]  

(11)

Now, using the result of [1], the optimal transformation that preserves the CRB for the whitened data set, \( \tilde{y}_w(t) \), should satisfy the condition \( \mathbb{R}(T_w) \supseteq \mathbb{R}(\widetilde{Q}^{-1/2}U) \). This condition is satisfied because \( B \) is full rank, see (1). Now, since \( y_1(t) = T_1^* \tilde{y}(t) = T_w \tilde{y}_w(t) \), we obtain the same CRB for estimates based on the data \( y_1(t) \) as for estimates based on elementspace data, \( \tilde{y}(t) \). The data \( y_2(t) \) can, in fact, be ignored since it conveys no more information about the parameters than is already present in \( y_1(t) \). We have thus shown that the beamspace CRB equals that of the elementspace provided that \( \mathbb{R}(T) \supseteq \mathbb{R}(\widetilde{Q}^{-1}U) \), which concludes the first part of the proof. It remains to show that \( \mathbb{R}(\widetilde{Q}^{-1}U) \supseteq \mathbb{R}(\widetilde{R}^{-1}U) \). The proof of this is based on the following two claims:

I. \( \widetilde{R}^{-1} \tilde{A} = \widetilde{Q}^{-1} \tilde{A} (I + R ss \tilde{A}^* \tilde{Q} \tilde{A})^{-1} \),

(12)

II. \( \widetilde{Q}^{-1} \tilde{D} = \widetilde{R}^{-1} \left[ \tilde{D} R ss \tilde{A}^* \tilde{Q} \tilde{A} \right] \)

(13)

**Proof of Claim I.** Multiplying the right-hand side by \( \widetilde{R} \), we have

\[
\widetilde{R}^{-1} \tilde{A} = \widetilde{Q}^{-1} \tilde{A} (I + R ss \tilde{A}^* \tilde{Q} \tilde{A})^{-1} = (\tilde{A} \tilde{R} ss \tilde{A}^* + \tilde{Q} \tilde{D} \tilde{R} ss \tilde{A}^* \tilde{Q} \tilde{A})^{-1} = \tilde{A} (R ss \tilde{A}^* \tilde{Q} \tilde{A} + I)^{-1} = \tilde{A}
\]

which proves I.

**Proof of Claim II.** Multiplying the right-hand side by \( \widetilde{Q} \), we get

\[
\widetilde{Q} \widetilde{R}^{-1} \left[ \tilde{A} \tilde{D} \right] R ss \tilde{A}^* \tilde{Q} \tilde{A} \tilde{D} = \tilde{D}
\]

(14)

The implication of Claim II is that

\[
\tilde{D} = \tilde{R}^{-1} \left[ \tilde{D} R ss \tilde{A}^* \tilde{Q} \tilde{A} \right] = \tilde{R}^{-1} U M_2
\]

where \( M_2 \) is the full-rank matrix defined by

\[
M_2 = \left[ \begin{array}{cc} R ss \tilde{A}^* \tilde{Q} \tilde{A} & \tilde{D} \\ I & \tilde{I} \end{array} \right].
\]  

(15)

Thus,

\[
\mathbb{R}(\widetilde{Q}^{-1}U) \supseteq \mathbb{R}(\widetilde{R}^{-1}U).
\]  

(16)

Using Eqs. (14) and (17), and noting that the range spaces of \( \widetilde{Q}^{-1}U \) and \( \widetilde{R}^{-1}U \) have the same dimension, yields the desired result

\[
\mathbb{R}(\widetilde{Q}^{-1}U) \supseteq \mathbb{R}(\widetilde{R}^{-1}U).
\]  

(18)

The proof of Theorem 1 is thereby complete. The extension of the result in Theorem 1 to the case where \( \theta \) contains several parameters, e.g., elevation, is straightforward. The matrix \( U(\theta_0) \) is modified to include the derivatives of the steering vector with respect to these parameters as well. The result in Theorem 1 also applies to other signal models, such as the deterministic parameterized signal model considered in [3].

There is a close similarity of the result in Claim I to the matched filter solution for detecting a single source at a known bearing in a colored noise environment. In that case, one can substitute the noise covariance matrix in the matched filter by the array covariance matrix without loss of detection performance, see, e.g., [7]. The explanation to this is that the matched filter is unique up to a multiplicative scalar, and since it holds that \( \widetilde{R}^{-1}a(\theta_0) \) is proportional to \( \tilde{Q}^{-1}a(\theta_0) \), the result follows. The result of Claim I can thus be seen as a generalization of the single-source case. However, the result in Claim II is more intricate and has no obvious analogy. It is the combination of the claims into a single condition on the range space of the transformation matrix that is important. Thus, the fact that the range space of \( \widetilde{R}^{-1}U(\theta_0) \) is the same as that of \( \tilde{Q}^{-1}U(\theta_0) \) is a key point in Theorem 1. If this was not true, an estimate of the noise covariance matrix \( \tilde{Q} \) would be needed. The array covariance matrix, however, can be estimated by a simple time average. In a real scenario, the beamspace transformation can be updated to
accommodate for a nonstationary scenario where both the source locations and the noise color may change with time. The matrix inversion lemma may be employed to recursively track the array covariance matrix in such a situation. An LMS-based update of the optimal beamspace transformation is also possible, for further reducing the computational complexity.

3 Implementing

We start this section by suggesting the following algorithm for beamspace DOA estimation:

1. Determine a set of interesting angle intervals that will be processed in beamspace. Define also the following quantity:

\[ U_f^{\text{DEF}} = [\tilde{a}(\theta_{f1}) \cdots \tilde{a}(\theta_{fn})] \]

where \( \tilde{a}(\theta_i) \), \( i = 1, \ldots, n \), is a sufficiently dense set of fictitious array steering vectors located within the selected angle intervals.

2. Estimate the array covariance matrix as

\[ \tilde{R} = \frac{1}{N} \sum_{i=1}^{N} \tilde{y}(t)\tilde{y}^*(t) + \varepsilon I \]

where \( \varepsilon I \) a possible regularization.

3. Calculate the quantity

\[ T_1 = \tilde{R}^{-1/2}U_fW \]

where \( W \) is a diagonal weighting matrix to be defined subsequently.

4. Decide on an initial beamspace dimension, denoted \( n_s \), that is large enough to accommodate for the maximum number of expected sources within the sectors.

5. Apply the singular-value decomposition, SVD, to the quantity in Eq. (20). The transformation is then created as

\[ T = \tilde{R}^{-1/2} [q_1 \cdots q_{n_s}] \]

where \( n_s \) is the number determined in Step 4 and \( q_i \), \( i = 1, \ldots, n_s \), are the corresponding left singular vectors.

6. Orthogonalize \( T \) (optional).

7. Optional step. Estimate the number of sources in the reduced beamspace domain using, e.g., the minimum description length (MDL), principle, see [9]. Do Step 4 again where now \( n_s \) is set to \( \min(n_s, 2 \hat{p}) \). Repeat Steps 5 and 6.

8. Estimate the directions of arrival.

Step 1 can preferably be based on the Capon spectrum, since this is used in the design of the beamspace transformation, see below. In this step, some kind of detection problem must be solved. A possible way to do this is just to look at the spatial spectrum and decide on which sectors to choose. Another way is the automated approach, in which a hypothesis problem can be posted, resulting in a thresholding problem. We do not treat this problem further in the present paper though. In Step 2, a possible regularization of the data covariance estimate is included. This regularization makes the method robust to estimation errors in the data covariance matrix when the number of samples is low. The choice of \( n_s \) in Step 4 may, e.g., be based on prior information on the maximum number of sources present in the sectors or such an estimate may be obtained by using, e.g., the MDL principle in elementspace. Another possibility is to choose the number \( n_s \) as the number of significant singular values in the SVD of Step 5. Step 7 facilitates a possible further reduction in the beamspace dimension, accompanied by a corresponding reduction of the computational load. Finally, we note that the reason why the derivatives of the fictitious steering vectors are not included in \( U_f \) is that these can be well approximated by a finite difference of the included fictitious steering vectors, implying that the range space of \( U_f \) approximates the range space of \( U(\theta_0) \).

The introduction of the weighting matrix in Step 3 is pertinent to the success of choosing the proper subspace dimension that the beamspace transformation shall preserve. We will now discuss two different choices of weighting matrices, \( W \). The first is the identity matrix, and the other is created by putting the Capon spectrum for the fictitious directions of arrival on the main diagonal. Starting with the identity matrix, a column of the matrix defined in Eq. (20) has the magnitude

\[ \left\| \tilde{R}^{-1/2} \tilde{a}(\theta_i) \right\|^2 = \tilde{\alpha}^* \tilde{R}^{-1/2} \tilde{R}^{-1/2} \tilde{\alpha} \]

which is exactly the inverse of the Capon spectrum at the point specified by the fictitious DOA. Thus, truncating the SVD with no weighting can result in noise subspace directions being chosen instead of the signal subspace directions, yielding a loss in performance. The second weighting is the Capon weighting. With this weighting, a column of the matrix in Eq. (20) has the magnitude

\[ \left\| \tilde{R}^{-1/2} \tilde{R}^{-1/2} \tilde{\alpha} \right\|^2 = \frac{1}{\tilde{\alpha}^* \tilde{R} \tilde{\alpha}} \]

which equals the Capon spectrum. This is the recommended weighting, since the problem of choosing the wrong subspace in the truncation of the SVD is eliminated.

4 Numerical examples

In this section we present computer simulations that support the theoretical result. First however, we briefly describe two approaches for finding a beamspace transformation matrix that reduce the dimensionality of the problem.
The author of [10] presents a method for calculating a near optimal, in the CRB sense, beamspace transformation matrix for the white-noise case. This method incorporates the use of a set of ‘design DOAs’ that cover an interval \([0^\circ, 0^\circ]\), within which the sources of interest are assumed to be located. The steering vectors of the design DOAs and the respective derivatives are collected in a matrix for which the singular-value decomposition (SVD), is calculated. The final transformation is then created by selecting as columns in the transformation matrix the left singular vectors corresponding to the \(n\) most significant singular values. As the singular vectors are orthogonal, the orthogonality constraint of the beamspace transformation matrix is satisfied. This method will be called the “white-noise method”, since it preserves the Cramér-Rao bounds for the spatially white-noise case. In [4], the first \(n\) spheroidal sequences are used as columns in the beamspace transformation matrix. The authors select \(n = \lfloor \frac{2Bm}{3} \rfloor - 2\), where \(\lfloor \cdot \rfloor\) denotes the integer part. The method supplies orthonormal beamformer weights that capture most of the energy of a spatially band-limited process with a flat energy spectrum within the sector \([0^\circ, 0^\circ]\), see [4, 8] for details. Finally, it should be noted that the parameter values used in Section 4 below result in \(n=2\) if the rule defined above is used. Since the initial beamspace dimension is chosen as 6 in Section 4, the resulting beamspace transformation matrix captures an amount of energy well above the limit implied by the rule above.

We employ a uniform linear array with 25 sensors, separated by half a wavelength. Two source signals are located at -70° and -40° within the interesting sector, defined by \([-10^\circ:10^\circ]\) relative broadside of the array. There are three sources located outside the sector, the first at 130°, the second at 180° and the location of the third is swept from -200° to -80° in steps of 20°. The sources outside the sector is referred to as ‘out-of-band’ sources. Note that the out-of-band sources act as interference, since we try to estimate only the direction of arrival of the sources within the interesting sector. In effect, we have a spatially colored noise problem. The initial beamspace dimension is set to 6, see Steps 4 and 7 of Section 3. The SNR values of the sources are equal and set to 10 dB relative to the noise power in one element. The number of snapshots is either 200 or 50 and the number of Monte Carlo trials is 100. A white-noise model is employed in beamspace together with the MDL estimate of the number of sources. The directions of arrival are estimated using the ‘stochastic maximum likelihood’ method, see, e.g., [6]. The numerical search is initiated in the following way: if the number of estimated sources in the MDL step is \(\hat{p}\), the search is initiated in the \(\hat{p}\) true DOAs within and nearest to the interesting sector. In Fig. 1, the root-mean-square error for the source located at -70° is plotted as a function of the location of the moving out-of-band source for the proposed method, with the two different weightings discussed in 3, and the methods of [4, 1, 8]. The number of snapshots is 200. No regularization of the estimated array covariance matrix is used in this case. As the number of samples is 200, the estimate of the array covariance matrix is of sufficient quality. In the figure, the solid line is the square root of the element space Cramer-Rao bound. The proposed method, with the proper weighting, yields an estimate with an accuracy very close to the bound. The other two methods, that have similar performance, produce estimates with considerably higher root-mean-square error. Note the severe degradation of the presented method if a proper weighting is not used. In Fig. 2, the approach is evaluated against the white-noise approach when the number of snapshots is decreased from 200 to 50. The result of this is that the quality of the array covariance estimate deteriorates, which in turn degrades the performance of the method. This is clearly seen from the result in Fig. 2. However, the good performance is restored by adding a scaled identity matrix to the estimated data covariance matrix making the method more robust to poor sample support (see Fig. 2). The excellent attenuation of the out-of-band sources for the proposed method results in that the estimated model order for this method is in general lower than for the other two methods.
5 Conclusions

This paper investigates the adaptive data reduction in sensor array processing for the colored noise case. The design criterion for the beamspace transformation is preservation of the Cramér-Rao bounds for the parameter estimates. A design procedure is given that produces a transformation that closely approximates the ideal one. The benefits are shown via computer simulations that focus on the problem of out-of-band sources. These can be viewed as interfering sources in beamspace, causing a loss in performance relative to elementspace estimation. The results indicate that significant improvements can be gained in terms of mean-square-error performance if the outlined approach is followed.

References