The Optimal Stopping Times of American Call Options

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Abstract: American options can be exercised at any time during their lifetime. This paper addresses the optimal stopping time of several kinds of American call options.

Key–Words: stopping time, American call option, martingale, equivalent martingale measure

1 Introduction

Option is a kind of financial derivative which came into being in the middle 1970s in America [1],[3]. As an efficient way to reduce risks, it has developed quickly since its emergence. According to transaction time, options can be divided into two sections: European options and American ones [4]. American options give the holder the right to exercise them at or before the expiry date, so the payoff of American options is determined by not only the price of underlying assets at maturity but also the price path. This property of American options makes it difficult to value them and determine optimal exercise moment [2].

In this paper, we consider the optimal stopping time and the price of American call options. We can show that the optimal stopping time of standard American option is their maturity in the model of continuous time, and the optimal stopping time of perpetual one does not exist. In the end we can also give the optimal stopping time of perpetual American call options when stock prices follow a jump-diffusion process and the initial price of them.

2 the Preliminary

Assumed that the financial market is composed of two kinds of assets. One of them, is called stock, whose price of per share $S(t)$ satisfies the equation

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t)$$

$$S(0) = s_0 > 0$$

Additionally, there is a risk-free asset, called the bond, whose price is given by

$$dB(t) = B(t)r(t)dt$$

$$B(0) = 1$$

Here $W(t)$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$, endowed with a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$, which is the P-augmentation of the natural filtration $\mathcal{F}^W(t) := \sigma(W(s) : 0 \leq s \leq t), 0 \leq t \leq T$, generated by $W(\cdot)$. The process $r(\cdot)$ (interest rate for lending), the process $\mu(\cdot)$ (return rate of the stock) and the process $\sigma(\cdot)$ (volatility of the stock) are all assumed to be progressively measurable with respect to $\mathcal{F}$. Moreover, the $\sigma(\cdot)$ is assumed to be positive, and all processes $r(\cdot), \mu(\cdot), \sigma(\cdot), \sigma^{-1}(\cdot)$ are assumed to be bounded, uniformly in $(t, \omega) \in [0, T] \times \Omega$.

Let

$$\theta(t) := \sigma^{-1}(t)(\mu(t) - r(t))$$

It is called the relative risk process of the market, and $\theta(t)$ is bounded and $\mathcal{F}$ progressively measurable, so by Itô formula, for any $0 \leq t \leq T$,

$$Z_0(t) := \exp \left\{-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds \right\},$$

is a martingale, and the process

$$W^0(t) := W(t) + \int_0^t \theta(s)ds, \quad 0 \leq t \leq T$$

is a Brownian motion under the probability measure

$$P^0(A) := E(Z_0(t); A), \quad A \in \mathcal{F}$$

Definition 1) An $\mathcal{F}$ progressively measurable process $\phi : [0, T] \times \Omega \rightarrow \mathbb{R}$ is called a portfolio process if $\int_0^T \phi^2(s)ds < \infty$ a.s.; 2) An $\mathcal{F}$ adapted process $C : [0, T] \times \Omega \rightarrow [0, \infty)$ is called a cumulative consumption process if $C$ is increasing, right continuous with $C(0) = 0, C(T) < \infty$ a.s.
Definition 2  1) A portfolio \( \phi(t) = \{\phi_0(t), \phi_1(t)\} \) is called a self-financing portfolio if the wealth process satisfies \( dV(t) = \phi_0(t)dB(t) + \phi_1(t)ds(t) \); 2) We say that a portfolio \( \phi(t) \) is admissible if the wealth process satisfies almost surely \( V_{\phi(t)}(t) \geq 0 \), for any \( 0 \leq t \leq T \).

Definition 3  1) A stopping time \( \tau^* \) is called the optimal exercised moment of a American option, if for any admissible self-financing portfolio \( \phi(t) \), \( V_{\phi(t)}(\tau^*) \geq f(\tau^*) \) \( P^0 - a.s., \) we always have,

\[
V_{\phi(t)}(\tau^*) = f(\tau^*).
\]

Here \( f(t) \) is a non-negative adaptive process with respect to the American contingent claim. If \( V_{\phi(t)}(t) \geq f(t), 0 \leq t \leq T, P^0 - a.s., \) we call the portfolio \( \phi(t) \) the duplicative portfolio of the contingent claim. Thus the meaning of the above definition is apparent. If there exists a set of positive probability such that \( V_{\phi(t)}(t, \omega) \geq f(t, \omega) \) when \( \omega \) is in the set, then the holder won’t exercise his/her right but choose to continue to hold it.

Lemma 1  Assumed that the stochastic process \( \phi(t) \) is a portfolio process and that a non-negative supermartingale \( Z(t) \) satisfies \( Z(t) = 1 + \int_0^t \phi(s)dW(t) \), if \( E(Z(t)) = 1, \) then

\[
W^*(t) = W(t) - \int_0^t Z^+(s)\phi(s)ds
\]

is a Brownian motion under \( Q \) probability measure. Here \( Q \) measure satisfies \( dQ = Z(T)dP, Z^+(t) = Z(t) \vee 0[5] \).

Lemma 2  Assumed that \( W(t) \) is a Brownian motion under \( P \) measure, for any \( \alpha \in R, \) let

\[
\tau_\alpha = \inf\{t \geq 0 : W(t) = \alpha\} = \inf\{t \geq 0 : W(t) \geq \alpha\},
\]

then \( P\{\tau_\alpha < \infty\} = 1 \). In addition, for any \( \alpha, \beta \in R, \) let

\[
\tau_{\alpha,\beta} = \inf\{t \geq 0 : W(t) = \alpha t + \beta\},
\]

then \( P\{\tau_{\alpha,\beta} < \infty\} = 1 \).

Proof  Let \( M(t) = \sup_{s \leq t}\{W(t)\}, \) we have

\[
F(t) = P\{\tau_\alpha \leq t\} = P\{M(t) \geq \alpha\} = 2P\{W(t) \geq \alpha\} = \frac{2}{\sqrt{2\pi t}} \int_\alpha^\infty e^{-x^2/2t}dx.
\]

Thus,

\[
P\{\tau_\alpha < \infty\} = \lim_{t \to \infty} F(t) = 1 - \lim_{t \to \infty} \frac{2}{\sqrt{2\pi t}} \int_\alpha^\infty e^{-x^2/2t}dx = 1.
\]

So let

\[
\frac{dQ}{dP} = Z^\alpha, \ Z^\alpha = \exp\{\mu W(t) - \frac{1}{2}\mu^2 t\},
\]

then \( W^*(t) = W(t) - \alpha t \) is a Brownian motion under \( Q \) measure, and \( \tau_{\alpha,\beta} = \tau^* = \inf\{t \geq 0 : W^*(t) = \beta\}, \) therefore,

\[
P\{\tau_{\alpha,\beta} < \infty\} = Q\{\tau^* < \infty\} = 1.
\]

In the following, \( \mu(t), r(t), \sigma(t) \) are abbreviated to \( \mu, r, \sigma \) respectively.

Lemma 3  There exists an equivalent martingale measure \( P^* \) of \( P \) measure such that the discount process \( \tilde{S}(t) = e^{-rt}S(t) \) \( 0 \leq t \leq T \) is a martingale under \( P^* \).

Proof  Let \( Z(t) = \exp\left\{-\frac{\mu - \sigma}{\sigma}Z(t) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right\}, \) then

\[
Z(t) > 0, P - a.s. \quad \text{and} \quad E(Z(t)) = 1, \quad \text{so according to Lemma 1,}
\]

\[
dP^* = Z(t)dP
\]

defines a probability measure on \( (\Omega, F_T) \). If we let \( W(t) = B(t) + \frac{\mu - \sigma}{\sigma}t, \) \( W(t) \) is a standard Brownian motion under \( P^* \). So,

\[
\frac{d\tilde{S}(t)}{\tilde{S}(t)} = -r e^{-rt}S(t)dt + e^{-rt}dS(t)
\]

(3)

\[
= \tilde{S}(t)[(\mu - r)dt + dB(t)] = \tilde{S}(t)\sigma dW(t).
\]

Thus \( \tilde{S}(t) = \tilde{S}(0) \exp\left\{\sigma W(t) - \frac{\sigma^2}{2} t\right\} \) is a martingale under \( P^* \).

3  the Optimal Stopping times

The valuation function of the American call option is \( U(X_t) = (X_t - k)^+ \), the holder always want to get the maximal profit, that is to say, he/she hopes to find the optimal exercise moment \( \tau^* \) such that \( C^*(0) = \sup_{\tau^*} E^*[e^{-rT}(S(\tau^*) - k)^+] = E^*[e^{-rT}(S(\tau^*) - k)^+]. \)

Theorem 1  i) The optimal stopping time of the American call option is \( T; \)

ii) The optimal stopping time of the permanent American call option does not exist. Let \( X(t) = \sigma W(t) + (r - \sigma/2)t, T_\epsilon = \frac{1}{r} \ln \left( \frac{k}{cS(0)} \right) \), \( 0 < \epsilon < k \), then \( T_\epsilon \) is the \( \epsilon \) optimal stopping time, and

\[
C^*(0)(1 - \epsilon) \leq E^*[e^{-rT_\epsilon}(S(T_\epsilon) - k)^+] \leq C^*(0).
\]
Proof  i) For \( S(t) = \exp \left\{ \sigma W(t) + \left( r - \frac{\sigma^2}{2} t \right) \right\} \), then \( e^{-rt} U(S(t)) = \exp \left\{ \sigma W(t) - \frac{\sigma^2}{2} t \right\} \), here \( \exp \left( \sigma W(t) - \frac{\sigma^2}{2} t \right) \) is a martingale, so by Jessen’s inequality, \( \left( \exp \left( \sigma W(t) - \frac{\sigma^2}{2} t \right) - k e^{-rt} \right)^+ \) is a sub-martingale. According to Doob’s stopping time theorem, for any \( \tau < T \), we have, 
\[
e^{-rt} U(S(\tau)) \leq e^{-rT} U(S(T))
\]
That’s, \( \tau^* = T \) is the optimal stopping time.
ii) The price process satisfies \( S(t) = \exp \left\{ \sigma W(t) + \left( r - \frac{\sigma^2}{2} t \right) \right\} \), so, 
\[
e^{-rt} U(S(t)) = e^{-rt} e^{\gamma X(t)} e^{-\gamma X(t)} U(S(t)).
\]
Let \( M(t) = e^{-rt} e^{\gamma X(t)} \), if \( M(t) \) is a martingale, \( \gamma \) must satisfies
\[\frac{1}{2} \gamma^2 \sigma^2 = \gamma (r - \frac{\sigma^2}{2}) - r,\]
that is, \( \gamma_1 = 1, \ \gamma_2 = -\frac{2r}{\sigma^2} \).

Therefore, by Girsanov theorem, \( M(t) = \exp \left\{ \gamma \sigma W(t) - \frac{1}{2} \gamma^2 \sigma^2 t \right\} \) is a martingale. Let \( M(0) = 1 \) and
\[f(x) = e^{\gamma x} (S(0) e^x - k), \ x \in (\ln \frac{k}{S(0)}, \infty)\]
If \( f'(x) = 0 \), \( x^* = \ln \frac{k}{S(0)} + \ln \frac{\gamma}{\gamma - 1} \). So \( f(x) \) attain its maximum \( \left( \frac{\gamma - 1}{k} \right)^{\gamma - 1} \gamma^{-\gamma} S(0)^{\gamma} \) at \( x = x^* \).

Denote
\[C^* = \left( \frac{\gamma - 1}{k} \right)^{\gamma - 1} \gamma^{-\gamma} S(0)^{\gamma} \]
Thus, for any \( t \geq 0 \),
\[e^{-rt} U(S(t)) = e^{-\gamma X(t)} U(S(t)) M(t) \leq C^* M(t).\]
Therefore, by Doob’s theorem of stopping time,
\[E \left[ e^{-rT} U(S(T)) I_{\{T < \infty\}} \right] \leq C^*\]
Especially,
\[E \left[ e^{-rT} U(S(\tau^*)) I_{\{\tau^* < \infty\}} \right] \leq C^* E(M(\tau^* I_{\{\tau^* < \infty\}})).\]

While \( \gamma_1 = 1, \ \gamma_2 = -\frac{2r}{\sigma^2} \), \( f'(x) > 0 \), so \( f(x) \) is a monotonically increasing function at interval \((\ln \frac{k}{S(0)}, \infty)\), and can’t attain its maximum, thus there doesn’t exist the optimal stopping time. As to \( T^* \),
\[E[e^{-rT^*} U(S(T^*))] = S(0) E \left[ e^{\sigma W(T^*) - \frac{1}{2} \sigma^2 T^* - ke^{-rT^*}} \right]^{+} = S(0) E \left[ e^{\sigma W(T^*) - \frac{1}{2} \sigma^2 T^* - ke^{-rT^*}} \right]^{+} \cdot I \{ \exp \{ \sigma W(T^*) - \frac{1}{2} \sigma^2 T^* \} - ke^{-rT^*} > 0 \} = S(0) E \left[ e^{\sigma W(T^*) - \frac{1}{2} \sigma^2 T^* - ke^{-rT^*}} \right]^{+} \cdot I \{ \exp \{ \sigma W(T^*) - \frac{1}{2} \sigma^2 T^* \} - ke^{-rT^*} < 0 \} \geq S(0) \left[ 1 - \frac{k}{S(0)} e^{-rT^*} \right] = S(0) (1 - \epsilon).\]

The desired conclusion is got.

In the following we consider a kind of American call options whose underlying asset’s prices follow a jump-diffusion process.

Assumed that the stock’s price process satisfies the following stochastic differential equation:
\[dS(t) = \mu(S(t)) dt + \sigma(S(t)) dW(t) + US(t) dN(t) \quad (4)\]
where \( N(t) \) is a Poisson process with parameter being \( \lambda \), \( U \) is a square integrable random variable, \( U > -1, P - a.s. \), additionally, \( W(t), N(t), U \) are independent.

The solution of equation (5) is,
\[S(t) = S(0) \exp \{ e^{\frac{\mu - \sigma^2}{2} t} + \sigma W(t) \} \prod_{n=1}^{N(t)} (1 + U_n),\]
where $U_1, U_2, \cdots$ are random variable with independent identical distribution function, $U_n$ is the jumping height of the underlying stock’s price at time $\tau_n$.

**Lemma 4** \{ $\phi_0(t), \phi_1(t)$ \} is self-financing if and only if

$$d\tilde{V}(t) = \phi_1(t)\tilde{S}(t)(\sigma dW^*(t) + UdN(t)).$$

where $W^*(t) = W(t) + \int_0^t \theta(s)ds$, $0 \leq t \leq T$.

By Itô’s Lemma and Girsanov Theorem, we can easily get it [7].

**Lemma 5** \[8\] If $E(|U_1|) < \infty$, the stock’s price process $\tilde{S}(t)$ with jump-diffusion is a martingale if and only if

$$\mu = r - \lambda E(U_1).$$

**Theorem 2** If the underlying stock’s price process is a one with Poisson jump, the optimal stopping time of the corresponding permanent American call option $f(X(t)) = \left\{e^{X(t)} \prod_{n=1}^{N(t)} (1 + U_n) - k\right\}^+$ is

$$\tau^* = \inf \left\{ t \geq 0 : \sigma W(t) = x^* - \left[ r - \frac{\sigma^2}{2} - \lambda E(U_1)\right] t \right\},$$

and the price of the option should be

$$C^* = e^{-\gamma_1 x^*} \left\{ e^{x^*} \prod_{n=1}^{N(t)} (1 + U_n) - k\right\}^+, \quad (5)$$

where $x^* = \ln \frac{\gamma_1 k}{\gamma - 1} - \sum_{n=1}^{N(t)} \ln(1 + U_n)$.

**Proof**

$C^* = \sup_{x \in R} \left\{ e^{\gamma x} f(x) \right\}$ can attain its extremum at

$$x^* = \ln k + \ln \frac{\gamma}{\gamma - 1} - \sum_{n=1}^{N(t)} \ln(1 + U_n).$$

Thus it should have $X(t) > \ln k - \sum_{i=1}^{N(t)} \ln(1 + U_i)$, that’s, $\gamma > 1$, so it must have $\gamma = \gamma_1$. Therefore the optimal stopping time is

$$\tau^* = \inf \left\{ t \geq 0 : X(t) = x^* \right\},$$

Noted that $X(t) = \sigma W(t) + \left[ r - \lambda E(U_1) - \frac{\sigma^2}{2}\right] t$ under an equivalent martingale measure, so

$$\tau^* = \inf \left\{ t : \sigma W(t) = x^* - \left[ r - \lambda E(U_1) - \frac{\sigma^2}{2}\right] t \right\}.$$}

By lemma 1, $P\{\tau^* < \infty\} = 1$, in addition, $C^* = \sup_{x \in R} \left\{ e^{\gamma x} f(x) \right\}$, thus,

$$C^* = e^{-\gamma_1 x^*} \left\{ e^{x^*} \prod_{n=1}^{N(t)} (1 + U_n) - k\right\}^+.$$

**References:**


