A Streamline Diffusion Method for a Singularly Perturbed Conservative Convection-diffusion Problem

ZHONGDI CEN  
Zhejiang Wanli University  
Institute of Mathematics  
Ningbo 315100  
P.R. China

LIFENG XI  
Zhejiang Wanli University  
Institute of Mathematics  
Ningbo 315100  
P.R. China

Abstract: In this paper we apply a streamline diffusion finite element method (SDFEM) to a singularly perturbed convection-diffusion problem. The stability and accuracy of the SDFEM on arbitrary grids are studied. We derive the pointwise error estimates and the approximation of derivatives. Numerical experiments support our theoretical results.

Key Words: Convection-diffusion, singular perturbation, streamline diffusion, Shishkin-type mesh

1 Introduction

Singularly perturbed differential equations arise frequently in many applied areas which include fluid dynamics, quantum mechanics, chemical reactions, and electrical networks. For the past two decades an extensive research has been made on numerical methods for the singularly perturbed differential equation, see [1,2] and reference therein.

It has been numerically observed that the streamline-diffusion finite element method (SDFEM) [3,4] often give a good and stable approximation of singularly perturbed boundary value problem if the grid is properly adapted to capture the singularity of the solution such as sharp layers. In this paper, we give a careful analysis of this phenomenon and develop a deeper understanding of the behavior of the SDFEM. The model problem we will study in this paper is a linear conservative convection-diffusion problem:

\[-\varepsilon u''(x) - (b(x)u(x))' = f(x), \; x \in (0, 1), (1)\]
\[u(0) = \gamma_0, \; u(1) = \gamma_1, \; (2)\]

where \(\varepsilon\) is a small positive parameter, \(b(x)\) and \(f(x)\) are sufficiently smooth, \(\gamma_0\) and \(\gamma_1\) are given constants, and for \(0 < x < 1\) we assume that \(b(x) \geq \beta > 0\). The solution \(u(x)\) of (1)-(2) typically has a boundary layer at \(x = 0\) and its derivatives can be bounded by

\[|u^{(k)}(x)| \leq C(1 + \varepsilon^{-k} \exp(-\beta x/\varepsilon)) \; (3)\]

for \(k = 0, 1, 2, 3, \; x \in [0, 1], \; \text{see [5]}.\)

The SDFEM, introduced first by Hughes and Brooks in [6], is one of such stabilized methods which combines good stability properties with high accuracy. In this paper we first analyze the SDFEM for the singularly perturbed problem (1)-(2) on arbitrary meshes. We derive the pointwise error estimates and the approximation of derivatives. These bounds are then made explicit for the Shishkin-type meshes.

2 Stability analysis of the SDFEM

In this section, we will study the stability of the SDFEM applied to equation (1)-(2) on arbitrary grids.

Let \(H^1 = \{v, v' \in L^2\}\) and \(H^1_0 = \{v | v \in H^1, v(0) = v(1) = 0\}\). The weak solution to (1)-(2) is a function \(u \in H^1\) satisfies \(u(0) = \gamma_0, u(1) = \gamma_1\) and

\[a(u, v) = (f, v), \; \forall v \in H^1_0, \; (4)\]

where \((\cdot, \cdot)\) is the \(L^2\) inner product and

\[a(u, v) = \varepsilon(u', v') + (bu, v') + \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_i} \delta_i(f - Lu)bv' dx,\]

where \(\delta_i\) is a stabilization function in \([x_{i-1}, x_i]\). We will discuss the choice of \(\delta_i\) later.

Here we assume that all integrals can be evaluated exactly. If this is not the case, then a suitable quadrature rule must be used. The existence and uniqueness of the weak solution are easy to establish.

For a positive integer \(N\), Let \(\Omega^N = \{x_i | 0 = x_0 < x_1 < \cdots < x_N = 1\}\) be an arbitrary grid. We denote by \(\varphi_i(x)\) the nodal basis function at point \(x_i\) and the
finite element space $V^N = \{ v^N = \sum_{i=0}^{N} a_i^N \phi_i(x) \}$. 
The finite element discretization of (4) is to find a $u^N \in V^N$ such that $u^N(0) = \gamma_0$, $u^N(1) = \gamma_1$ and

$$a(u^N, v^N) = (f, v^N), \quad \forall v^N \in V^N \cap H^0_\delta.$$  (5)

Let $e(x) = (u^t - u^N)(x) = \sum_{i} e_i^N \phi_i$ with $e_i = e(x_i)$, $i = 1, 2, \cdots, N-1$, where $u^t$ denote the piecewise linear interpolation on the given mesh. Since $a(u - u^N, v^N) = 0$, we obtain the error equation

$$a(e, \varphi_i) = a(u^t - u, \varphi_i), \quad i = 1, 2, \cdots, N - 1, \quad (6)$$

$$e_0 = e_{N+1} = 0.$$ (7)

Let $a_{ij} = a(\varphi_j, \varphi_i)$ and $h_i = x_i - x_{i-1}$. A routine calculation shows that for $i = 1, 2, \cdots, N - 1$

$$a(e, \varphi_i) = a_{i-1,i} e_{i-1} + a_{i,i} e_{i} + a_{i+1,i} e_{i+1} + f_i - \bar{f}_{i+1},$$

where

$$a_{i,i-1} = -\frac{\varepsilon}{h_i} + \delta_i b_{i-1/2} b_{i-1/2},$$

$$a_{i,i} = \frac{\varepsilon}{h_i} + \delta_i b_{i-1/2} - \delta_i b_{i+1/2} + \frac{\delta_i b_{i-1/2} + \delta_i b_{i+1/2}}{2},$$

$$a_{i,i+1} = -\frac{\varepsilon}{h_i} - \frac{b_{i+1/2}}{h_i} - \delta_i b_{i+1/2} b_{i+1/2},$$

$$f_i = h_i^{-1} \int_{x_{i-1}}^{x_i} \delta_i f(x) b(x) dx,$$

$$b_{i-1/2} = b(x_{i-1} + x_i/2).$$

When $\varepsilon$ is small relative to the local meshsize, a standard way of stabilizing this scheme is to choose $\delta_i$ according to the formula $\delta_i = h_i/(2b_{i-1})$. If the local meshsize is small enough—in particular, if $b_{i-1/2} b_{i} < 2\varepsilon$—then the standard Galerkin method works well, so it is possible to choose $\delta_i = 0$. Thus, to stabilize the scheme, we choose

$$\delta_i = \begin{cases} 0 & \text{if } b_{i-1/2} h_i < 2\varepsilon, \\ h_i/(2b_{i-1}) & \text{if } b_{i-1/2} h_i \geq 2\varepsilon. \end{cases}$$ (8)

**Lemma 1.** The error equation (6)-(7) can be written as

$$A^N e_i - A^N e_{i+1} = r_i - r_{i+1}, \quad 1 \leq i < N, \quad (9)$$

$$e_0 = e_{N+1} = 0, \quad (10)$$

where

$$A^N e_i = \left( \frac{\varepsilon}{h_i} + \frac{b_{i-1/2}}{2} + \frac{\delta_i}{h_i} b_{i-1/2} \right) e_i - \left( \frac{\varepsilon}{h_i} - \frac{b_{i-1/2}}{2} + \frac{\delta_i}{h_i} b_{i-1/2} \right) e_{i-1}$$ (11)

and

$$r_i = h_i^{-1} \int_{x_{i-1}}^{x_i} (u^t - u)(x) b(x) dx$$

$$+ \int_{x_{i-1}}^{x_i} \delta_i u'' b(x) dx$$

$$- \int_{x_{i-1}}^{x_i} \delta_i (u^t - u)(x)' b(x) dx.$$ (12)

**Proof.** Clearly,

$$A^N e_i - A^N e_{i+1} = a_{i+1,i} e_{i+1} - a_{i,i} e_{i} + a_{i,i+1} e_{i+1} + (\bar{f}_i - \bar{f}_{i+1}) = a(u^t - u, \varphi_i) - (\bar{f}_i - \bar{f}_{i+1}).$$

Note that $\int_0^1 (u^t - u') \varphi_i'(x) dx = 0$,

$$a(u^t - u, \varphi_i) = \int_{x_{i-1}}^{x_i} b(x)(u^t - u) \varphi_i(x) dx$$

$$+ \int_{x_{i-1}}^{x_i} \delta_i (f - L(u^t - u)) b(x) \varphi_i(x) dx$$

$$+ \int_{x_{i-1}}^{x_i} \delta_i (f - L(u^t - u)) b(x) \varphi_i'(x) dx$$

$$= r_i - r_{i+1} + (\bar{f}_i - \bar{f}_{i+1})$$

and the desired result follows from this.

It is easy to see that $A^N e_i = r_i + C$ with an appropriate constant $C$ such that $e_0 = e_{N+1} = 0$. However it is difficult to determine $C$ explicitly. Instead we use the following splitting of $e_i$.

$$e_i = W_i - \frac{V_i}{V_N} W_N,$$ (13)

where $V$ is the solution of the difference equation

$$A^N V_i = 1, \quad i = 1, 2, \cdots, N, \quad V_0 = 0,$$

and $W$ is the solution of the difference equation

$$A^N W_i = r_i, \quad i = 1, 2, \cdots, N, \quad W_0 = 0.$$ (14)

**Proof.** It is clear that $e_i = W_i - CV_i$. Since $e_N = 0$, we get $C = W_N/V_N$. The matrix associated with $A^N$ is a bidiagonal M-matrix. Consequently one can use suitable barrier functions and the definitions of $\{V_i\}$ and $\{W_i\}$ to show that

$$0 \leq V_i \leq 1, \quad W_i \leq \|r\|\infty V_i$$ (14)

for $i = 0, 1, 2, \cdots, N$. Thus, we have

$$|e_i| \leq |W_i| + \frac{|W_i|}{V_N} V_i \leq 2 \|r\|\infty$$ (15)
for \( i = 1, 2, \ldots, N \).

Furthermore,

\[
A^N e_i = A^N W_i - \frac{W_N}{V_N} A^N V_i = r_i - \frac{W_N}{V_N}. \tag{16}
\]

From (16) and (14) we have

\[
|A^N e_i| \leq 2\|r\|_{\infty} \quad \text{for} \quad i = 1, 2, \ldots, N. \tag{17}
\]

Since

\[
A^N e_i = \varepsilon D^{-} e_i + b_{i-1/2} e_i + e_{i-1} \frac{e_{i+1} - e_{i-1}}{2} + \delta b_{i-1/2} e_i - b_{i-1} e_{i-1} h_i,
\]

we obtain

\[
\varepsilon |D^{-} e_i| \leq C \|r\|_{\infty} \quad \text{for} \quad i = 1, 2, \ldots, N, \tag{18}
\]

where we have used (17) and (15).

Now we can bound the pointwise errors in the computed solution and the \( \varepsilon \)-weighted errors \( \varepsilon D^{-} (u_i - u_i^N) \).

**Theorem 1.** There exist constants \( C \) such that

\[
|u_i - u_i^N| + \varepsilon |D^{-} (u_i - u_i^N)| \leq C \|r\|_{\infty} \tag{19}
\]

for \( i = 1, 2, \ldots, N \).

**Proof.** From (15) we have

\[
|u_i - u_i^N| = |u_i^I - u_i^N| \leq C \|r\|_{\infty} \tag{20}
\]

for \( i = 1, 2, \ldots, N \).

Similarly, from (18) we have

\[
\varepsilon |D^{-} (u_i - u_i^N)| \leq \varepsilon |D^{-} (u_i - u_i^I)| + \varepsilon |D^{-} e_i| \leq C \|r\|_{\infty} \tag{21}
\]

for \( i = 1, 2, \ldots, N \).

Combining (20) with (21), we get the desired results.

### 3 Analysis on Shishkin-type meshes

In this section let \( N \) be an even integer. We shall consider a mesh \( \Omega^N \) that is equidistant in \([x_{N/2}, 1]\) but graded in \([0, x_{N/2}]\), where we choose the transition point \( x_{N/2} \) as Shishkin does:

\[
x_{N/2} = \tau = \frac{2\varepsilon}{\beta} \ln N. \tag{22}
\]

On \([0, x_{N/2}]\) let our mesh be given by a mesh-generating function \( \varphi \), with \( \varphi(0) = 0 \) and \( \varphi(1/2) = \ln N \), where \( \varphi \) is continuous, monotonically increasing and piecewise continuously differentiable. Then our mesh is

\[
x_i = \begin{cases} \frac{2\varepsilon}{\beta} \varphi(t_i) & t_i = i/N, \\ 1 - (1 - \frac{2\varepsilon}{\beta} \ln N)2(N-1)N, & N/2 < i \leq N. \end{cases}
\]

We define a new function \( \psi \) by \( \psi(t) = \exp(-\varphi(t)), t \in [0, 1/2] \). This function is monotonically decreasing with \( \psi(0) = 1 \) and \( \psi(1/2) = N^{-1} \). Examples of the mesh-characterizing function \( \psi \) are

\[
\psi(t) = 1 - 2(1 - N^{-1})t
\]

for Bakhvalov-Shishkin mesh and

\[
\psi(t) = e^{-2(\ln N)t}
\]

for standard Shishkin mesh.

For Shishkin-type meshes we have the following general result [7].

**Lemma 3.** Let us assume that the mesh-generating function \( \varphi \) is piecewise differentiable and that it satisfies the condition

\[
\max_{x \in [0,1/2]} \varphi'(x) = \max_{x \in [0,1/2]} |\varphi'| \leq C N. \tag{23}
\]

Then

\[
\vartheta_k(\Omega^N) = \max_{i=1,\ldots,N} \int_{x_{i-1}}^{x_i} \left[ 1 + e^{-\varepsilon \exp(-\beta x/(k\varepsilon))} \right] dx 
\leq C \{ \varepsilon + N^{-1} \max_{x \in [0,1/2]} |\varphi'(x)| \} \tag{24}
\]

for \( k = 1, 2, \ldots \).

The following interpolation error estimate for Shishkin-type meshes is well known; see for example [8].

**Lemma 4.** Assume that the piecewise differentiable mesh generating function \( \varphi \) satisfies (23). Then the interpolation error for linear interpolation on the Shishkin-type meshes satisfies

\[
|(u - u^I)(x)| \leq \begin{cases} C(N^{-1} \max |\varphi'|)^2, & x \in [0, x_{N/2}], \\ C N^{-2}, & x \in [x_{N/2}, 1]. \end{cases}
\]

The next lemma gives us a useful estimate for \( r_i \) on Shishkin-type meshes.

**Lemma 5.** Assume that the condition (23) holds true. Then on Shishkin-type meshes we have

\[
|r_i| \leq C(N^{-1} \max |\varphi'|)^2 \quad \text{for} \quad i = 1, 2, \ldots, N.
\]
Proof. From (12) we have
\[
|r_i| \leq |h_i^{-1} \int_{x_{i-1}}^{x_i} (u' - u)(x) b(x) dx| \\
+ |h_i^{-1} \int_{x_{i-1}}^{x_i} \delta(\epsilon u''(x) b(x) dx| \\
+ |h_i^{-1} \int_{x_{i-1}}^{x_i} \delta(b(x)(u' - u)(x)) b(x) dx| \\
\leq C \max_{x_{i-1} \leq x \leq x_i} |(u' - u)(x)| \\
+C \delta_i h_i^{-1} \int_{x_{i-1}}^{x_i} (1 + e^{-2} \exp(-\beta x/\epsilon)) dx \\
+C \delta_i h_i^{-1} [b_(u' - u)(x_i) \\
- b_(u' - u)(x_i-1)] \\
\leq C \max_{x_{i-1} \leq x \leq x_i} |(u' - u)(x)| + C \epsilon \delta_i \\
+C \delta_i h_i^{-1} \exp(-\beta x_{i-1}/\epsilon),
\]
where we have used (3).

Thus, using the lemma 4 and (8) we obtain
\[
|r_i| \leq C (N^{-1} \max |\psi'|)^2 \text{ for } i = 1, 2, \cdots, N,
\]
where we have used (22).

With the interpolation error estimates, we can get the convergence approximation of the SDFEM.

**Theorem 2.** Assume that the condition (23) holds true. Then on Shishkin-type meshes we have the following error estimates:
\[
|u_i - u_i^N| + \epsilon |D^- (u_i - u_i^N)| \\
\leq C (N^{-1} \max |\psi'|)^2, \quad (25)
\]
and
\[
\max_{x_{i-1} \leq x \leq x_i} \epsilon |D^- u_i^N - u_i'(x)| \leq C N^{-1} \max |\psi'|, \quad (26)
\]
\[
\epsilon |D^- u_i^N - u_i'(x_{i-1}/2)| \leq C (N^{-1} \max |\psi'|)^2. \quad (27)
\]

**Proof.** The first result follows immediately from Theorem 1 and Lemma 5.

Next, using a Taylor expansion for \( u \) and \( u' \) about \( x_i \), we get
\[
\max_{x_{i-1} \leq x \leq x_i} \epsilon |D^- u_i - u_i'(x)| \leq C \epsilon \int_{x_{i-1}}^{x_i} |u''(x)| dx \\
\leq C \int_{x_{i-1}}^{x_i} (1 + \epsilon^{-1} \exp(-\beta x/\epsilon)) dx \\
\leq N^{-1} \max |\psi'|,
\]
where we have used (3) and (24). Combining this inequality with the first result, we obtain the second result.

Finally, we use Taylor expansions for \( u \) and \( u' \) about \( x_i \) to obtain
\[
\epsilon \left| \frac{u_i - u_i-1}{h_i} - u_i'/2 \right| \\
\leq \frac{3\epsilon}{2} \int_{x_{i-1}}^{x_i} |u''(t)|(t - x_{i-1}) dt \\
\leq \frac{3\epsilon}{2} \int_{x_{i-1}}^{x_i} (t - x_{i-1})(1 + \epsilon^{-3} \exp(-\beta t/\epsilon)) dt
\]
by (3). To bound the right-hand side we use the inequality in [9]
\[
\int_{x_{i-1}}^{x_i} g(\xi)(\xi - x_{i-1}) d\xi \leq \frac{1}{2} \left\{ \int_{x_{i-1}}^{x_i} g(\xi)^{1/2} \right\}^2
\]
which holds true for any positive monotonically decreasing function \( g \) on \([x_{i-1}, x_i]\). This can be easily verified by considering the two integrals as functions of the upper integration limit. We get
\[
\epsilon \left| \frac{u_i - u_i-1}{h_i} - u_i'/2 \right| \\
\leq C \left\{ \int_{x_{i-1}}^{x_i} (1 + \epsilon^{-1} \exp(-\beta t/(2t))) dt \right\}^2 \\
\leq C (N^{-1} \max |\psi'|)^2
\]
by (24). Combining this inequality with the first result, we obtain the third result.

## 4 Numerical experiments

In this section we verify experimentally the theoretical results obtained in the preceding section.

**Example** Consider the problem
\[
-\epsilon u''(x) - ((1 + x)u(x))' = f(x), \quad 0 < x < 1, \quad (28)
\]
\[
u(0) = u(1) = 0, \quad (29)
\]
where \( f(x) \) is chosen such that
\[
u(x) = \frac{1 - \exp(-x/\epsilon)}{1 - \exp(-1/\epsilon)} - x
\]
is the exact solution.

For our tests we take \( \epsilon = 10^{-8} \) which is a sufficiently small choice to bring out the singularly perturbed nature of the problems. In order to evaluate the integrals in (5), we apply the standard midpoint rule
\[
\int_{x_{j-1}}^{x_j} \Psi(x) x \sim (x_j - x_{j-1})\Psi(x_{j-1}/2).
\]

We measure the accuracy of the pointwise error estimates and the approximation of derivatives in the
Table 1: The pointwise error estimates of the SDFEM on the standard Shishkin mesh for Example 2

| N   | ||u − u^N||_∞ error rate | rate |
|-----|------------------------|------|
| 32  | 4.8669e-3              | 1.441|
| 64  | 1.7928e-3              | 1.501|
| 128 | 6.3324e-4              | 1.581|
| 256 | 2.1163e-4              | 1.636|
| 512 | 6.8100e-5              | 1.678|
| 1024| 2.1286e-5              | -     |

Table 2: The approximation of derivatives of the SDFEM on the standard Shishkin mesh for Example

<table>
<thead>
<tr>
<th>N</th>
<th>max_{i=1,···,N-1} ε</th>
<th>D^-u^N_i - u(x_i)</th>
<th>error rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>1.7220e-1</td>
<td>0.602</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>1.1348e-1</td>
<td>0.695</td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>7.0111e-2</td>
<td>0.758</td>
<td></td>
</tr>
<tr>
<td>256</td>
<td>4.1444e-2</td>
<td>0.802</td>
<td></td>
</tr>
<tr>
<td>512</td>
<td>2.3772e-2</td>
<td>0.832</td>
<td></td>
</tr>
<tr>
<td>1024</td>
<td>1.3354e-2</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

The numerical results (Tables 1-6) are clear illustrations of the convergence estimate of Theorem 2. They indicate that the theoretical results are fairly sharp.

Acknowledgements: The research was supported by Zhejiang Province Natural Science Foundation (grant No. Y607504) of China and Ningbo Natural Science Foundation (grant No. 2007A610048) of China.

References:


