The Solving of a Stationary Transport Equation by
Fictitious Domain Method

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Abstract: In this paper a numerical method for solving a two-dimensional stationary transport equation is presented. Using the techniques of the variational calculus and method of the fictitious domain, we create an algorithm that leads to the solution of a boundary – value problem for the set \( D \), which has an arbitrary shape. In this approach, the initial domain \( D \) (called physical domain) is immersed in a square domain \( D_2 \). The theoretical results are verified with the help of a numerical example.

Key-Words: transport equation, integral-differential equation, fictitious domain method, variational method

1 Introduction

The problems that deal with the transport of neutral particles in a scattering and fission event with no self-interactions are the subject of several papers. The authors of these have proposed a variety of numerical methods: the splitting method,[1],[6], the quadrature formulas for discretizations of angular variables, the finite differences method [3], [4], [5], the finite element, [1],[2], the Monte Carlo method, Nyström method and spectral methods, [4],[7]–[9]. Also, the \( F_N \) method is a semi-analytical method suitable for obtaining highly accurate solutions for the transport problems in spherical and cylindrical geometries.

In this work, we propose an algorithm that allows the reducing a boundary – values problem for a two-dimensional transport equation to a Dirichlet problem for an elliptic partial differential equation. The difficulties appear when the solution is defined in a domain \( D \) that has an arbitrary shape. By embedding of \( D \) into square \( D_2 \) and using the techniques of the fictitious domain method, [3],[4], we obtain a new equation that contains additional terms and new boundary conditions.

Then, using a numerical example, we verify the convergence of the solutions obtained by our algorithm, when the steps of network for the spatial domain tend to zero.

2 Variational method

Let us consider that the neutrons move to a direction, which makes the angle \( \alpha \) with Ox axis and the angle \( \beta \) with Oy axis. We denote \( \mu = \cos \alpha \) and \( \nu = \cos \beta \). The integral-differential equation of the transport theory for a two-dimensional geometry in a stationary case is of the form

\[
\mu \frac{\partial \varphi}{\partial x}(x, y, \mu, \nu) + \nu \frac{\partial \varphi}{\partial y}(x, y, \mu, \nu) + \varphi(x, y, \mu, \nu) = \int \int_{D_1} \varphi(x, y, \mu', \nu') d\mu' d\nu' + f(x, y, \mu, \nu)
\]

(1)

with the boundary condition

\[
\varphi(x, y, \mu, \nu) \bigg|_{\partial D_2} = 0, \quad \partial D_2 - \text{ boundary of } D_2 \quad (2)
\]

where \( \varphi(x, y, \mu, \nu) \) represents the flux of neutrons by a surface of the unit sphere \( S_1 \) and \( f(x, y, \mu, \nu) \) is a given radioactive source function.

Let \( \varphi \) be a continuous function on \( D_1 \) and with the continuous second derivatives on \( D_2 \). In
order to solve the problem (1) - (2), we add the equations obtained by partial derivatives of (1) with respect to the variables \(x\) and \(y\), multiplied by \(\mu\) and \(\nu\) respectively. As a result, we get

\[
\mu^2 \frac{\partial^2 \varphi}{\partial x^2} + 2\nu \frac{\partial^2 \varphi}{\partial x \partial y} + \nu^2 \frac{\partial^2 \varphi}{\partial y^2} - \varphi = F(x, y, \mu, \nu)
\]

where

\[
F(x,y,\mu,\nu) = \mu \left[ \frac{\partial \varphi}{\partial x}(x, y, \mu', \nu')d\mu'd\nu' + \right. \\
\left. + \nu \left[ \frac{\partial \varphi}{\partial y}(x, y, \mu', \nu')d\mu'd\nu' - \int \frac{\partial \varphi}{\partial y}(x, y, \mu, \nu) - f(x, y, \mu, \nu) \right] \right]
\]

Next, we present a variational method for the solving of problem (3) with the boundary condition (2). For this, we consider a square grid \(\Delta_1\) for \(D_1\) with the step \(k\): \(\Delta_1 = \{(\mu_i, \nu_j) \in D_1 | i \in \{0,...,N\}, j \in \{0,...,N\}\}\)

and another square grid for \(\Delta_2\):

\[
\Delta_2 = \{(x_i, y_j) \in D_2 | i \in \{0,..., N+1\}, j \in \{0,..., N+1\}\}
\]

with the step \(h\). As shown in Figure 1 for \(N = 3\), each square of the \(h\) side from \(D_2\) will be split into two isosceles triangles. In view of [6], let us consider the approximate solution of the equation (3) of the form (4):

\[
\varphi^h(x, y) = \sum_{m,n=1}^N \alpha_{mn}(\mu, \nu) \omega_{mn}(x, y)
\]

If

\[
\alpha_{mn}(\mu, \nu) = \nu + \eta_m \mu + \eta_n \nu
\]

where \(\eta_m\) and \(\eta_n\) are the constants, the function \(F\) becomes

\[
F(x,y,\mu,\nu) = \mu \left[ \frac{\partial f(x,y,\mu,\nu)}{\partial x} + \nu \left[ \frac{\partial f(x,y,\mu,\nu)}{\partial y} - f(x,y,\mu,\nu) \right] \right]
\]

Now, we shall determine the solution of equation (3) for every fixed values \((\mu_k, \nu_j)\)\(\in\Delta_1\). Let us denote

\[
u(x, y) = \varphi(x, y, \mu, \nu)\]

and (4) becomes

\[
u(x, y) = \sum_{m,n=1}^N \alpha_{mn}(\mu, \nu) \omega_{mn}(x, y)
\]

We shall find \(\alpha_{mn}\) such that (6) to minimize the following functional

\[
J(u) = \int_{D_2} \left[ \mu^2 \left( \frac{\partial u}{\partial x} \right)^2 + 2\mu \nu \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \nu^2 \left( \frac{\partial u}{\partial y} \right)^2 + u^2 \right] dxdy + 2 \int_{D_2} F(x,y) dxdy
\]

The pyramidal functions \(\omega_{mn}(x, y)\) correspond to the \(N^2\) hexagonal regions, \(D_{mn}\), which are centered at the points \(P_{mn}\) of the network \(\Delta_2\). Each hexagonal subdomain is the union of six triangles, \(\{D_{mn,j}\}, j \in \{1,2,...,6\}\), the numbering of which is shown in Fig.1. The geometric interpretation of the function \(\omega_{mn}\) is a pyramid with a height equals to unity and with the projection \(P_{mn}\) of its apex (Fig.2). We define the function \(\omega_{mn}\) of the following form

\[
\omega_{mn}(x, y) = \left\{ \begin{array}{ll}
1 - \frac{1}{h} (x_m - x) - \frac{1}{h} (y_n - y), & \text{if } (x, y) \in D_{mn,1} \\
1 - \frac{1}{h} (x_m - x), & \text{if } (x, y) \in D_{mn,2} \\
1 + \frac{1}{h} (y_n - y), & \text{if } (x, y) \in D_{mn,3} \\
1 + \frac{1}{h} (x_m - x) + \frac{1}{h} (y_n - y), & \text{if } (x, y) \in D_{mn,4} \\
1 + \frac{1}{h} (x_m - x), & \text{if } (x, y) \in D_{mn,5} \\
1 - \frac{1}{h} (y_n - y), & \text{if } (x, y) \in D_{mn,6} \\
0, & \text{in rest.}
\end{array} \right.
\]

Minimization of the functional (7) leads to a system of \(N^2\) equations with the \(\alpha_{mn}\) unknowns
$$\frac{\partial J(\phi^b)}{\partial \alpha_{il}} = 0, \quad k, l \in \{1,2,\ldots,N\}$$

The matrix form of this system is

$$A \cdot \alpha = g$$

and solving it we get $\alpha$. If the equation (3) is rewritten in the form

$$\sum_{s,t=1}^{2} a_{st} \frac{\partial^2 u}{\partial x_s \partial x_t} - u = F_1$$

where $x_1 = x, x_2 = y, a_{11} = \mu_1^2, a_{12} = a_{21} = \mu_\nu, a_{22} = \nu_1^2$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N^2})^T$ (T - transpose matrix) is the unknown vector for $k, l \in \{1, 2, \ldots, N\}$:

$$\varphi(x_k, y_j, \mu_\nu, \nu_\nu) = u(x_k, y_j) = \alpha_{lk}$$

for every fixed point $(\mu_\nu, \nu_\nu) \in \Delta_i$. The elements $\gamma_{ij}$ of matrix $A$ form a band along its main diagonal and are defined by the following formula

$$\gamma_{N(k-1)+l, N(m-1)+n} = \int \int_{D_{kl}} \int \int_{D_{kl}} a_{st} \frac{\partial \omega_{mk}}{\partial x_s} \frac{\partial \omega_{ml}}{\partial x_t} + \omega_{l1} \omega_{mn} dx dy$$

where $k, l, m, n = 1, 2, \ldots, N$.

The vector $g = (g_1, g_2, \ldots, g_{N^2})$ from (9) has the components of the form

$$g_{N(k-1)+l} = \int \int_{D_{kl}} F_1 \omega_{kl} dx dy, \quad k, l = 1, 2, \ldots, N$$

3 Fictitious domain method

Let us now consider the equation (10) for every fixed point $(\mu_\nu, \nu_\nu) \in \Delta_i$ and $(x, y) \in D \subset D_2$ (Fig.3):

$$a_{11} \frac{\partial^2 u}{\partial x^2} + 2a_{12} \frac{\partial^2 u}{\partial x \partial y} + a_{22} \frac{\partial^2 u}{\partial y^2} - u = F_1$$

In an operatorial form, the equation (17) becomes

$$Lu = F_1$$

and the boundary condition is

$$u(x, y) = 0, \forall (x, y) \in \partial D$$

Next, we replace the problem (14) - (15) with a new boundary value problem defined on the square $D_2$ that contains the initial domain $D$. If $D_2 = D \setminus D$, we denote the common boundary of $D$ and $D_2$ by $S$ (Fig. 3).

![Diagram](image_url)

The equation (14) becomes

$$b_{11} \frac{\partial^2 v}{\partial x^2} + 2b_{12} \frac{\partial^2 v}{\partial x \partial y} + b_{22} \frac{\partial^2 v}{\partial y^2} - v = F_2$$

or

$$\frac{\partial v}{\partial x} \left( b_{11} \frac{\partial v}{\partial x} + b_{12} \frac{\partial v}{\partial y} + b_{22} \frac{\partial v}{\partial y} \right) - v = F_2$$

where

$$b_{ij} = \begin{cases} a_{ij}, & (x, y) \in D \\ 0, & (x, y) \in D_2, i \neq j \\ \frac{1}{\epsilon^2}, & (x, y) \in D_3, i = j \end{cases}$$

$$F_2(x, y) = \begin{cases} F_1(x, y), & (x, y) \in D \\ 0, & (x, y) \in D_3 \end{cases}$$

the number $\epsilon$ being very little with respect to the unity. The form (18) of the coefficients $b_{ij}$ is the consequence of the following physical property: in a domain with the great diffusive coefficient, the density of the diffusible matter varies too little. The operatorial form of (14) is now of the form

$$L_1 v = F_2$$

and the boundary conditions that accompany (20) are

$$v(x, y)|_{\partial D_2} = 0, \quad [v(x, y)]_S = 0$$

and

$$\left[ \left( b_{11} \frac{\partial v}{\partial x} + b_{12} \frac{\partial v}{\partial y} \right) \cos(\bar{n}, O\nu) + b_{22} \frac{\partial v}{\partial y} \cos(\bar{n}, O\nu) \right]_S = 0$$

where $[ \ ]$ is the jump of $v$ and of the partial derivatives at the crossing of $S$. The vector $\bar{n}$ is the normal to $S$.

4 Numerical example

Let us consider the transport equation

$$\mu \frac{\partial \varphi}{\partial x} (x, y, \mu, \nu) + v \frac{\partial \varphi}{\partial y} (x, y, \mu, \nu) + \varphi(x, y, \mu, \nu) = f(x, y, \mu, \nu)$$

where

$$(x, y) \in D = [1/3, 1] \times [0, 1], \quad (\mu, \nu) \in D_1 = [-1, 1] \times [-1, 1]$$

and

$$f = \mu \nu \left( \frac{3\pi}{2} \sin \pi y - \frac{3\pi}{2} \sin \frac{3\pi}{2} \sin \frac{3\pi}{2} \sin \pi y + \frac{3\pi}{2} \sin \frac{3\pi}{2} \sin \pi y + \frac{3\pi}{2} \sin \frac{3\pi}{2} \sin \pi y + \right)$$
\[ \phi(x,y,\mu,\nu)|_{\partial D} = 0, \partial D \text{ - boundary of } D \]  

(23)

The boundary condition is of the form

\[ \varphi(x,y,\mu,\nu) = \frac{3\pi x}{2} \]  

(27)

where

\[ B = B_1 + B_2 \]

The boundary conditions that accompany (25) coincide with (21).

We shall determine the approximate solution of the problem (22)-(23) in the nodes (21) and (22) using the fictitious domain method (Fig. 4). Let us immerse the domain \( D_2 = [0, 1] \times [0, 1] \) and equation (22) becomes

\[ b_{11} \frac{\partial^2 \psi}{\partial x^2} + 2b_{12} \frac{\partial^2 \psi}{\partial x \partial y} + b_{22} \frac{\partial^2 \psi}{\partial y^2} - \psi = F_2 \]  

(24)

where \( b_{ij} \) and \( F_2 \) are defined by (18) and (19), respectively. If we use the notations from (10), the equation (24) can be rewritten in the form

\[ \sum_{i,j=1}^{2} b_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi = F_2 \]  

(25)

The boundary conditions that accompany (25) will be of the form (6)

\[ \psi^h(x,y,\mu,\nu) = \sum_{m,n=1}^{2} \alpha_{mn} \omega_{mn}(x,y) \]  

(26)

Next, we shall denote \( \psi^h \) by \( \psi \) for the easiness of writing. It follows from the equation (10) that the matrix \( A \) becomes the matrix

\[ A = B_1 + B_2 \]  

(27)

The elements of matrix \( B \) were computed by (11). For example, when \( k = 1, l = 1, m = 1, n = 2 \) we get

\[ \gamma_{12} = \int_0^{1/3} \int_{2/3-x}^{2/3} \left( \sum_{i,j=1}^{2} b_{ij} \frac{\partial \omega_{i1}}{\partial x_i} \frac{\partial \omega_{j1}}{\partial x_j} - \omega_{i1} \omega_{j1} \right) dx dy \]

which correspond to \( D_{11,3}, D_{11,4}, D_{12,1}, D_{12,6} \) and

\[ g = \begin{bmatrix} 0 \\ 0 \\ g_{21} \\ g_{22} \end{bmatrix}, \quad g_{ij} = -\int_{D_2} F_2(x,y,\mu_i,\nu_j) \omega_{ij} dx dy \]

Then, the solution in extended domain \( D_2 \) is

\[ \bar{\psi} = B^{-1} g \]  

(28)

The exact solution of our problem (22)-(23) is

\[ \phi(x,y,\mu,\nu) = \mu \nu \cos \frac{3\pi x}{2} \sin \pi x \sin \pi y \]  

(29)

In the following tables 1-3 are presented the values of \( v(x_k, y_k, \mu, \nu_j) \) (the step \( h = 1/3 \)) and \( v\psi(x_k, y_k, \mu, \nu_j) \) (the step \( h = 1/6 \)) computed with the help of a program written in MathCAD, where \((x_k, y_l)\) correspond to the nodes of (21) and (22), respectively for every fixed \((\mu, \nu_j)\).

The approximate solutions \( v \) and \( v\psi \) are compared with the exact solution \( \psi(x_k, y_k, \mu, \nu_j) = \phi(x_k, y_k, \mu, \nu_j) \).
### TABLE I. Comparison between numerical and exact solution for \( \mu = -3/4 \) and \( \mu = -1/2 \)

<table>
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<th>( \nu )</th>
<th>( \nu \nu )</th>
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### TABLE II. Comparison between numerical and exact solution for \( \mu = -1/4 \) and \( \mu = -1/4 \)

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### TABLE III. Comparison between numerical and exact solution for \( \mu = -1/2 \) and \( \mu = 3/4 \)

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<td>( \nu = 1/2 )</td>
<td>-0.135</td>
<td>-0.17</td>
<td>-0.188</td>
<td>( \nu = 1/2 )</td>
<td>-0.206</td>
</tr>
<tr>
<td>( \nu = 3/4 )</td>
<td>-0.222</td>
<td>-0.268</td>
<td>-0.281</td>
<td>( \nu = 3/4 )</td>
<td>-0.345</td>
</tr>
<tr>
<td>( \nu = 3/4 )</td>
<td>-0.195</td>
<td>-0.26</td>
<td>-0.281</td>
<td>( \nu = 3/4 )</td>
<td>-0.3</td>
</tr>
</tbody>
</table>
Figures 5, 6 show the variation of the numerical solution \( v^* \) (red color) when \( h = 1/6 \) and exact solution \( v_x \) (green color) that correspond to the nodes (21) and (22) for the different values of \((\mu, \nu)\in\Delta_1 \times \Delta_1\).

5 Conclusions

The results obtained in the numerical example prove that the approximation corresponding to the fictitious domain method is of the order of \( \varepsilon \).

A study of the errors shows that there is different values \( v \) for the nodes (21) and (22) at the same \((\mu, \nu)\in\Delta_1\). The difference depends on the step \( h \) and in view of [3] this is of the \( h^2 \) order for the function \( \varphi \in C^2(D_2) \). It follows from the values of \( v \) and \( v^* \) presented in the tables I – III and shown in the figures 5, 6 that the numerical solutions converge to the exact solution for \( h \) tends to zero.

On the other hand, the errors increase with the increasing of module of \( \mu \) and \( \nu \) and decrease with the decreasing of the step \( h \).

References: