Fuzzy Multiobjective Bimatrix Games: introduction to the computational techniques

ANDRE A. KELLER
Université de Haute Alsace, Mulhouse, FRANCE
andre.keller@uha.fr

Abstract: This paper introduces to the computational techniques of non-cooperative bimatrix games in an uncertain environment. Both single and multiple objective fuzzy-valued bimatrix games are considered theoretically with one numerical example. The presentation is centered on the Nishizaki and Sakawa models. These models are based on the maxmin and minmax principles of the classical matrix game theory. Equivalent nonlinear (possibly quadratic) programming problems are giving optimal solutions. The equilibrium solutions correspond to players trying to maximize a degree of attainment of the fuzzy goals. The aggregation of all the fuzzy sets in the multiobjective models use the fuzzy decision rule by Bellman and Zadeh. This "aggregation by a minimum component" consists in the intersection of the fuzzy sets, the fuzzy expected payoffs and the fuzzy goals. Numerical examples of two-players nonzero sum games are solved using the Mathematica software. The numerical solutions are possibly using iterative methods.

Key–Words: Fuzzy bimatrix game, single-objective, multiobjective, degree of attainment of a fuzzy goal, quadratic programming

1 Introduction

The non-cooperative bimatrix games in the literature differ according two main aspects: the number of the objectives and the type of fuzziness of the goals and payoffs of the game. Indeed, the bimatrix games may be with a single objective or with multiple objectives (Nishizaki and Sakawa [10], 2000; Chen [4], 2002). The uncertainty may also concern the goals or the payoffs (Bector and Chandra [1], 2005; Wang et al. [13], 2007; Han et al. [5], 2009; Larbani [6], 2009) or both goals and payoffs (Nishizaki and Sakawa [10, 11], 2000, 2001; Vidyottama et al. [16], 2004). This presentation considers both aspects.

1.1 Single objective bimatrix game

Two players I and II have mixed strategies given by the \(n\)-dimensional vector \(x\) and the \(m\)-dimensional vector \(y\), respectively. Let \(e_n\) be an \(n\)-dimensional vector of ones, \(e_m\) having a dimension \(m\). Suppose that the strategy spaces of Player I and II are defined by the convex polytopes \(S^n = \{x \in \mathbb{R}^n_+, x e_m = 1\}\) and \(S^m = \{y \in \mathbb{R}^m_+, y e_n = 1\}\), respectively. The payoffs of Players I and II are the \(m \times n\) matrices \(A\) and \(B\), respectively. The objectives of Player I and Player II will be the programming problems \(\max x' Ay\) subject to \(x' e_m = 1, x \geq 0\), and \(\max y' By\) subject to \(y' e_n = 1, y \geq 0\), respectively. The expected payoffs of Players I and II are

\[E_1(x,y) = x' Ay\] \[E_2(x,y) = x' By\]

respective. Playing safe, the two players will select the strategy for which the maximum losses are minimum.

1.2 Equilibrium solution

Definition 1 A Nash equilibrium point is a pair of strategies \((x^*, y^*)\) such that the objectives of the two players are full filled simultaneously. We have

\[x^*Ay^* = \max_x \{x' Ay | x' e_m = 1, x \geq 0\}\]
\[x^*By^* = \max_y \{x' By | y' e_n = 1, y \geq 0\}\]

Applying the Kuhn-Tucker necessary and sufficient conditions, we have the Equivalence Theorem 2.

Theorem 2 (Mangasarian and Stone (1964)[7])

(Equivalence Theorem) Let \(G = (S^n, S^m, A, B)\) be a bimatrix game, a necessary and sufficient condition that \((x^*, y^*)\) be an equilibrium point is the solution of the QP problem

\[
\max_{x,y,p,q} x'(A + B)y - p - q \\
\text{subject to} \\
Ay \leq pe_n, \\
B'x \leq qe_m, \\
x' e_m = 1, \\
y' e_n = 1, \\
x \geq 0, y \geq 0,
\]
where \( p, q \in \mathbb{R} \) are the negative of the multipliers associated with the constraints.

**Proof:** see appendix A. \( \square \)

The Lemke-Howson’s algorithm (1964) [8, 9, 17] can be used when computing the Nash equilibrium payoffs.

### 1.3 Bimatrix game with fuzzy goals

**Definition 3** Let the expected payoff of Player I be \( D_1 = \{x'A| x \in S^m, y \in S^n\} \). A fuzzy goal for Player I is a fuzzy set \( G_1 \) represented by the membership function \( \mu_1 : D_1 \rightarrow [0, 1] \).

**Definition 4** Let the expected payoff of Player II be \( D_2 = \{x'By| x \in S^m, y \in S^n\} \). A fuzzy goal for Player II is similarly a fuzzy set \( G_2 \) represented by the MF \( \mu_2 : D_2 \rightarrow [0, 1] \).

An equilibrium solution is defined with respect to (w.r.t.) the degree of attainment of the fuzzy goals.

**Definition 5** A pair \( (x^*, y^*) \in S^m \times S^n \) is an equilibrium solution if, for other strategies, we have

\[
\begin{align*}
\mu_1(x'A y^*) & \geq \mu_1(x'A y_0^*), \text{ for all } x \in S^m \\\\\\text{ and } y \in S^n, \\
\mu_2(x'B y^*) & \geq \mu_2(x'B y_0^*), \text{ for all } y \in S^n \\
\end{align*}
\]

The expression of the linear MF of the fuzzy goal \( G_1 \) for Player I is

\[
\mu_1(x'A) = \begin{cases} 
1, & x'A \geq \bar{a} \\
\frac{x'A - a}{\bar{a} - a}, & a < x'A < \bar{a} \\
0, & x'A \leq a,
\end{cases}
\]

where \( a \) denotes the worst degree of satisfaction of Player I, whereas \( \bar{a} \) denotes the best degree of satisfaction. These values are defined as

\[
\begin{align*}
a = \min_{x \in X} \min_{y \in Y} x'A \\
\bar{a} = \max_{x \in X} \max_{y \in Y} x'A
\end{align*}
\]

The expression of the linear MF of the fuzzy goal \( G_2 \) for Player II is, as well

\[
\mu_2(x'B) = \begin{cases} 
1, & x'B \geq \bar{b} \\
\frac{x'B - b}{\bar{b} - b}, & b < x'B < \bar{b} \\
0, & x'B \leq b,
\end{cases}
\]

where \( b \) and \( \bar{b} \) also denote the worst and the best degree of satisfaction of Player II, respectively. These values are deduced from similarly using \( B \).

**Theorem 6** (Equilibrium solution) An equilibrium solution \( (x^*, y^*) \) of the fuzzy bimatrix game, is deduced from the optimal solution \( (x^*, y^*, p^*, q^*) \) of the QP problem

\[
\max_{x^*,y^*,p,q} x' (A + B) y - p - q
\]

subject to

\[
\begin{align*}
Ay & \leq pe_n, \\
B'x & \leq qe_m, \\
x'e_m & = 1, \\
y'e_n & = 1, \\
x & \geq 0, & y & \geq 0,
\end{align*}
\]

where \( \hat{A} = A/(\bar{a} - a) \) and \( \hat{B} = B/(\bar{b} - b) \).

**Proof:** see Bector and Chandra [1], p. 180. \( \square \)

### 1.4 Numerical example

In the Nishizaki and Sakawa’s multiobjective example ([11], pp. 93–95), Player I has three pure strategies and Player II four strategies. Let us retain a single objective version, with the following payoffs

\[
A = \begin{pmatrix} 1 & 4 & 7 & 2 \\ 3 & 6 & 1 & 8 \\ 2 & 5 & 3 & 9 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & 1 & 2 & 4 \\ 3 & 4 & 8 & 3 \\ 1 & 8 & 1 & 2 \end{pmatrix}
\]

The values of the worst and the best degree of satisfaction are given by \( a = b = 1, \bar{a} = 9, \bar{b} = 8 \). We have the QP problem

\[
\max_{x,y,p,q} \frac{1}{8} \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 4 & 7 & 2 \\ 3 & 6 & 1 & 8 \\ 2 & 5 & 3 & 9 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \]

\[
+ \frac{1}{7} \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 & 4 \\ 3 & 4 & 8 & 3 \\ 1 & 8 & 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \]

\[-p - q \]

subject to

\[
\begin{align*}
\frac{1}{8} \begin{pmatrix} 1 & 4 & 7 & 2 \\ 3 & 6 & 1 & 8 \\ 2 & 5 & 3 & 9 \end{pmatrix} & \leq p \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\
\frac{1}{7} \begin{pmatrix} 5 & 3 & 1 \\ 1 & 4 & 8 \\ 2 & 8 & 1 \\ 4 & 3 & 2 \end{pmatrix} & \leq q \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\
x_1 + x_2 + x_3 & = 1, \\
y_1 + y_2 + y_3 + y_4 & = 1,
\end{align*}
\]

\[
x' = (x_1, x_2, x_3) \geq 0, \quad y' = (y_1, y_2, y_3, y_4) \geq 0.
\]
The optimum solutions of the QP problem have been obtained by an iterative method. We have \( x^* = (0.4795, 0.2877, 0.2329), \ y^* = (0.0481, 0.5948, 0.2857, 0.0714) \), \( p^+ = 0.5712 \), \( q^* = 0.4990 \).

\[ \text{Definition 7 } \text{The fuzzy decision is expressed in terms of the intersection of the fuzzy expected payoffs and the fuzzy goals} \]

\[ \mu_{a(x,y)} = \min \left\{ \mu_{xG_y}(p), \mu_{G_y}(p) \right\}, \]

where \( p \) is a payoff for Player I. A degree of attainment of the fuzzy goal is defined as the maximum of the MF \( \mu_{a(x,y)} \). We have

\[ d_1(x, y) = \max \min \left\{ \mu_{xG_y}(p), \mu_{G_y}(p) \right\}. \]

The degree of attainment of the fuzzy goal for Player II \( d_2(x, y) \) is similarly defined.

\[ \text{Definition 8 } \text{For any pair of strategies} \ (x, y), \text{the degree of attainment of the fuzzy goal for Player I and Player II are respectively} \]

\[ d_1(x, y) = \frac{x(A + \Delta A)y - a}{a - a + x\Delta A y} \]

\[ d_2(x, y) = \frac{x(B + \Delta B)y - b}{b - b + x\Delta B y}, \]

where \( \Delta A \) (resp. \( \Delta B \)) denotes the right spread matrix of the fuzzy matrix \( A_{LR} \) (resp. \( B_{LR} \)).

\[ \text{2.2 Equilibrium solution} \]

\[ \text{Definition 9 Let} \ G = (S^m, S^n, \tilde{A}, \tilde{B}) \text{be a fuzzy bimatrix game, the equilibrium solution} \ w.r.t. \text{the degree of attainment of the fuzzy goal is a pair of strategies} \ (x^*, y^*) \text{if, for all other strategies, we have} \]

\[ d_1(x^*, y^*) \geq d_1(x, y^*) \text{for all} \ x \in S^m \]

\[ d_2(x^*, y^*) \geq d_2(x^*, y) \text{for all} \ y \in S^n \]

The programming problem of the Player I is

\[ \max_x \ d_1(x, y^*) = \frac{x(A + \Delta A)y^* - a}{a - a + x\Delta A y^*} \text{subject to} \]

\[ x^Te_m = 1, \]

\[ x \geq 0. \]

The programming problem of the Player II is

\[ \max_y \ d_2(x^*, y) = \frac{x^*(B + \Delta B)y - b}{b - b + x^*\Delta B y} \text{subject to} \]

\[ y^Te_n = 1, \]

\[ y \geq 0. \]

\[ \text{Nishizaki and Sakawa} \ [10] \text{also consider a convex combination.} \]
Applying the Kuhn–Tucker necessary and sufficient conditions, we have the equivalence Theorem 10:

**Theorem 10 (Equivalence Theorem)** Let $G = (S^m, S^n, \tilde{A}, \tilde{B})$ be a fuzzy bimatrix game, a necessary and sufficient condition that $(x^*, y^*)$ be an equilibrium point, is the solution of the non linear programming problem

$$\begin{align*}
\max_{x, y, \psi, \xi} & \quad a^T x' (A + \Delta_A) y + b^T y' (B + \Delta_B) y \\
\text{subject to} & \quad (a - a + x' \Delta_A) y + (a - x' A)(\Delta_A) y \\
& \quad -\psi(a - a + x' \Delta_A) y \leq 0, \\
& \quad (a - a + x' \Delta_A) y + (a - x' A)(\Delta_A) y \\
& \quad -\psi(a - a + x' \Delta_A) y \leq 0, \\
& \quad (b - b + x' \Delta_B) y + (b - x' B)(\Delta_B) y \\
& \quad -\xi(b - b + x' \Delta_B) y \leq 0, \\
& \quad (b - b + x' \Delta_B) y + (b - x' B)(\Delta_B) y \\
& \quad -\xi(b - b + x' \Delta_B) y \leq 0, \\
& \quad x'e_m = 1, \\
& \quad y'e_n = 1, \\
& \quad x \geq 0, \quad y \geq 0,
\end{align*}$$

where $\psi, \xi$ are scalars, $A_i$ and $B_i$, $i = 1, 2$ are the $i$th row of matrices $A$ and $B$, respectively.

**Proof:** see Nishizaki & Sakawa \(^4\) [11], pp. 105–108

\[ \square \]

### 2.3 Numerical example

In the following two players example \(^5\), Players I and II have two pure strategies. The goals of the two players are fuzzy. The payoffs are triangular fuzzy numbers. The LR-representations of the payoffs are the tensors $\tilde{A} \in \mathbb{R}^{2 \times 2 \times 3}$ and $\tilde{B} \in \mathbb{R}^{2 \times 2 \times 3}$ for Players I and II, respectively are

$$\begin{align*}
\tilde{A}_{LR} &= \begin{pmatrix} (180, 5, 10) & (156, 6, 2) \\
(90, 10, 10) & (180, 5, 10) \end{pmatrix} \\
\tilde{B}_{LR} &= \begin{pmatrix} (200, 10, 15) & (132, 4, 6) \\
(120, 5, 10) & (156, 6, 10) \end{pmatrix}
\end{align*}$$

The right spread matrices are

$$\begin{align*}
\Delta_A &= \begin{pmatrix} 10 & 2 \\
10 & 10 \end{pmatrix} \\
\Delta_B &= \begin{pmatrix} 15 & 6 \\
10 & 6 \end{pmatrix}
\end{align*}$$

The optimal solutions \(^6\) of Player I are $x_{1}^* = 0.2466$ and $x_{2}^* = 0.7534$ w.r.t. a degree of attainment of the goal \(^7\) of 75.4 per cent. The optimal solutions of Player II are $y_{1}^* = 0.2963$ and $y_{2}^* = 0.7037$ w.r.t. a degree of attainment of the goal of 39.2 per cent.

### 3 Multiobjective fuzzy bimatrix game

A multiple objectives bimatrix game is considered in a fuzzy environment where both the objectives and the payoffs are uncertain. The list of the $r$ payoff matrices for Player I is represented by $\tilde{A} = (\tilde{a}_{ij})_{m \times n}, \ k \in \mathbb{N}_r$. The list of the $s$ payoff matrices for Player II is represented by $\tilde{B} = (\tilde{b}_{ij})_{m \times n}, \ l \in \mathbb{N}_s$.

#### 3.1 Fuzzy expected payoff

For triangular fuzzy numbers, we have the LR-representations of entries $\tilde{a}_{ij} = (a_{ij}, \delta_{ij}^{-}, \delta_{ij}^{+})_{LR}$ and $\tilde{b}_{ij} = (b_{ij}, \delta_{ij}^{-}, \delta_{ij}^{+})_{LR}$.

**Definition 11** For any pair of mixed strategies $(x, y)$, the $k$th fuzzy expected payoff of Player I is defined by

$$x^A_k y = (x' A^k y, x' \Delta_A^{k-} y, x' \Delta_A^{k+} y)_{LR}$$

and is characterized by the MF

$$\mu_{x^A_k y} : D_k \mapsto [0, 1].$$

The $l$th fuzzy expected payoff of Player II is similarly defined by

$$x^B_l y = (x' B^l y, x' \Delta_B^{l-} y, x' \Delta_B^{l+} y)_{LR}$$

and is characterized by the MF

$$\mu_{x^B_l y} : D_l \mapsto [0, 1].$$

\(^4\)The notations are those of this article except for the right spread matrices which are denoted by $\tilde{A}$ and $\tilde{B}$ in [11].

\(^5\)This numerical application is an extension of the Campos’s example [3].

\(^6\)The numerical solutions have been obtained using the primitive ‘Minimize’ of the software Mathematica, with a computing time of 23 minutes.

\(^7\)We have $d_1^* = \frac{x^*(A + \Delta_A) y - \bar{y}}{x^* (A + \Delta_A) y} = 0.7540$. 
3.2 Fuzzy goal attainment

Definition 12 Let the fuzzy goals of Players I and II be denoted by \( p_1 = (p_1^1, \ldots, p_1^n) \in D_1 \subseteq \mathbb{R}^r \) and \( p_2 = (p_2^1, \ldots, p_2^n) \in D_2 \subseteq \mathbb{R}^s \). The Player I's \( k \)th fuzzy goal \( G_1^k \) is a fuzzy set characterized by the MF

\[
\mu_1^k : D_1^k \mapsto [0,1].
\]

Similarly, the Player II's \( l \)th fuzzy goal \( G_2^l \) is a fuzzy set characterized by the MF

\[
\mu_2^l : D_2^l \mapsto [0,1].
\]

Definition 13 For any pair of strategies \((x,y)\), the an attainment state of the fuzzy goal is represented by the intersection of the fuzzy expected payoff \( \tilde{x} \tilde{A} \) \( y \) and the fuzzy goal \( G_1^k \). We have

\[
\mu_{a(x,y)}(p) = \min \left\{ \mu_{a(x,y)}^k(p), \mu_{G_1^k}(p) \right\},
\]

where \( p \in D_1^k \) is a payoff of Player I. The degree of attainment of the \( k \)th fuzzy goal for Player II is the maximum of the MF, such as

\[
\hat{\mu}_{a(x,y)}^k(p^*) = \max_p \mu_{a(x,y)}^k(p).
\]

Similarly, the degree of attainment of the fuzzy goal for Player II is

\[
\hat{\mu}_{b(x,y)}^l(p^*) = \min_p \mu_{b(x,y)}^l(p).
\]

3.3 Equilibrium solution

An equilibrium solution is defined w.r.t. the degree of attainment of the aggregated fuzzy goal.

Definition 14 Let \( G = (S^m, S^n, \tilde{A}, \tilde{B}) \) be a fuzzy bimatrix game, the equilibrium solution w.r.t. the degree of attainment of the aggregated fuzzy goal is a pair of strategies \((x^*, y^*)\) if, for all other strategies, we have

\[
D^1(x^*, y^*) \geq D^1(x, y^*), \text{ for all } x \in S^m
\]

\[
D^2(x^*, y^*) \geq D^2(x^*, y), \text{ for all } y \in S^n.
\]

If the fuzzy goals are aggregated by a minimum component, the classical decision rule by Bellman and Zadeh [2] is used \(^8\). This aggregation method consists in the intersection of all the fuzzy sets. The Player I's degree of attainment of the aggregated fuzzy goal is defined by

\[
D^1(x, y) = \min_{k \in \mathbb{N}} \frac{x'(A^k + \Delta^k_A)y - a^k}{\bar{a}^k - a^k + x'\Delta^k_A y}.
\]

The mathematical programming problem of the Player I is

\[
\max_{x, \sigma} \quad \sigma
\]

subject to

\[
\frac{x'(A^k + \Delta^k_A)y^* - a^k}{\bar{a}^k - a^k + x'\Delta^k_A y^*} \geq \sigma,
\]

\[
x'e_m = 1,
\]

\[
x \geq 0.
\]

The mathematical programming problem of the Player II is

\[
\max_{y, \delta} \quad \delta
\]

subject to

\[
\frac{x^*(B^l + \Delta^l_B)y - b^l}{b^l - b^l + x^*\Delta^l_B y} \geq \delta,
\]

\[
y'e_n = 1,
\]

\[
y \geq 0.
\]

Applying the Kuhn-Tucker necessary and sufficient conditions, we have the equivalence Theorem 15.

Theorem 15 (Equivalence Theorem) Let \( G = (S^m, S^n, \tilde{A}, \tilde{B}) \) be a multiobjective fuzzy bimatrix game, a necessary and sufficient condition that \((x^*, y^*)\) be an equilibrium point is the solution of
the nonlinear programming problem

\[
\max_{x, y, \psi, \xi, \sigma, \Delta, \Theta} \left\{ \sum_{k=1}^{r} \lambda_k \left[ \frac{a^k (2x' \Delta^k_A y + \bar{a}^k - a^k)}{(\bar{a}^k - a^k + x' \Delta^k_A \bar{y})^2} - x' \Delta^k_A y \times x' (A^k + \Delta^k_A) y \right] + \sigma - \psi \\
+ \sum_{l=1}^{s} \theta_l \left[ \frac{b^l (2x' \Delta^l_B y + \bar{b}^l - b^l)}{(\bar{b}^l - b^l + x' \Delta^l_B \bar{y})^2} - x' \Delta^l_B y \times x' (B^l + \Delta^l_B) y \right] + \delta - \xi \right\}
\]

subject to

\[
\sum_{k=1}^{r} \lambda_k \left[ \frac{a^k - a^k + x' \Delta^k_A y}{(\bar{a}^k - a^k + x' \Delta^k_A \bar{y})^2} \right] - \psi \leq 0,
\]

\[
\sum_{k=1}^{r} \lambda_k \left[ \frac{a^k - a^k + x' \Delta^k_A y A^k y}{(\bar{a}^k - a^k + x' \Delta^k_A \bar{y})^2} \right] - \psi \leq 0,
\]

\[
\sum_{l=1}^{s} \theta_l \left[ \frac{(\bar{b}^l - b^l + x' \Delta^l_B y) (B^l y)}{(\bar{b}^l - b^l + x' \Delta^l_B \bar{y})^2} \right] - \xi \leq 0,
\]

\[
\sum_{l=1}^{s} \theta_l \left[ \frac{(\bar{b}^l - b^l + x' \Delta^l_B y) (B^l y)}{(\bar{b}^l - b^l + x' \Delta^l_B \bar{y})^2} \right] - \xi \leq 0,
\]

\[
\frac{x' (A^k + \Delta^k_A y - a^k)}{\bar{a}^k - a^k + x' \Delta^k_A \bar{y}} - \sigma \geq 0, \quad k \in \mathbb{N}_r
\]

\[
\frac{x' (B^l + \Delta^l_B y - b^l)}{\bar{b}^l - b^l + x' \Delta^l_B \bar{y}} - \delta \geq 0, \quad l \in \mathbb{N}_s
\]

\[
x' \epsilon_m = 1, \quad y' \epsilon_n = 1,
\]

\[
x \geq 0, \quad y \geq 0, \quad \Lambda \geq 0, \quad \Theta \geq 0,
\]

where \(\psi, \xi\) are scalars and \(\Lambda' = (\lambda_k)_{1 \times 3}, \Theta' = (\theta_l)_{1 \times 3}\), scalar entries. The vector \(A^k_i, i = 1, 2\) denotes the \(i\)th row of the matrix \(A^k\) and similarly for \(B^l_j, j = 1, 2, 3\).

**Proof:** see Nishizaki & Sakawa [11], pp. 110–114 \(\square\)

### 3.4 Numerical example

In the following two player example 9. Players I and II have respectively two and three pure strategies and three different objectives. The goals of the two players are fuzzy. The payoffs are triangular fuzzy numbers. The LR-representation of the payoffs are the tensors \(A^k \in \mathbb{R}^{2 \times 3 \times 3}, k \in \mathbb{N}_3\) and \(\bar{B} \in \mathbb{R}^{2 \times 2 \times 3}, l \in \mathbb{N}_3\) for Players I and II respectively, are

\[
\bar{A}^1_{LR} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad \bar{A}^2_{LR} = \begin{pmatrix} 4 & 1 & 5 \\ 1 & 1 & 1 \end{pmatrix}, \quad \bar{A}^3_{LR} = \begin{pmatrix} 2 & 1 & 5 \\ 0 & 1 & 5 \end{pmatrix}
\]

and

\[
\bar{B}^1_{LR} = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 1 & 5 \end{pmatrix}, \quad \bar{B}^2_{LR} = \begin{pmatrix} 4 & 5 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \bar{B}^3_{LR} = \begin{pmatrix} 2 & 1 & 5 \\ 0 & 1 & 5 \end{pmatrix}
\]

The right spread matrices for Player I are

\[
\Delta^1_A = \begin{pmatrix} 1 & 1 & 1.5 \\ 1.5 & 1 & 1 \end{pmatrix}, \quad \Delta^2_A = \begin{pmatrix} 1 & 1 & 0.5 \\ 1.5 & 1 & 1 \end{pmatrix}
\]

\[
\Delta^3_A = \begin{pmatrix} 1.5 & 1.5 & 1 \\ 1.5 & 1.5 & 1 \end{pmatrix}
\]

The right spread matrices for Player II are

\[
\Delta^1_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0.5 \end{pmatrix}, \quad \Delta^2_B = \begin{pmatrix} 1 & 1.5 & 0.5 \\ 1 & 1.5 & 1 \end{pmatrix}
\]

\[
\Delta^3_B = \begin{pmatrix} 0.5 & 1 & 1.5 \\ 0.5 & 1.5 & 1 \end{pmatrix}
\]

The optimal solutions 10 of Player I are \(x^*_1 = .6438\) and \(x^*_2 = .3562\) w.r.t. a degree of attainment of the

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9 This numerical application is an extension of the Chen’s example [4].

10 The numerical solutions have been obtained using the primitive ‘Minimize’ of Mathematica for a calculation time of 7 minutes 46.
goal of 58.5 cent. The optimal solutions of Player II are \( y_1^* = .5226, y_2^* = .3149 \) and \( y_3^* = .1625 \) w.r.t. a degree of attainment of the goal of 52.5 per cent.

## 4 Conclusion

The crisp bimatrix games have an equivalent QP problem for finding Nash equilibrium solutions. The single objective fuzzy bimatrix game have an equivalent nonlinear programming problem. The multiple objective bimatrix games have an extended nonlinear programming problem. All these problems may be solved by different ways, by using algorithms and optimization techniques (Lemke-Howson’s algorithm, multipliers, Van de Panne’s two phase method [14]), genetic algorithm [13], the relaxation procedure for min-max problems subject to separate constraints (see appendix B), introducing Nature as a third Player [6].

### A Proof of the Theorem 2

The equilibrium solution can be obtained by solving [4, 7]

\[
\max_{x,y} x' Ay + x' B y \\
\text{subject to} \\
x' e_m = 1, \\
y' e_n = 1, \\
x \geq 0, y \geq 0,
\]

Let \( p = \max_x x' Ay \) and \( q = \max_x x' B y \). The following constraint is also true \( p \geq x' Ay \geq x' By \), for all \( x \geq 0 \). So we have the simplification \( p e_m \geq Ay \). We also have the constraint \( q \geq x' B y \), for all \( y \geq 0 \). So we have the simplification \( q e_n' \geq B' x \). The QP problem is

\[
\min_{x,y,p,q} (p - x' Ay) + (q - x' By) \\
\text{subject to} \\
A y \leq p e_m, \\
B' x \leq q e_n', \\
x' e_m = 1, \\
y' e_n = 1, \\
x \geq 0, y \geq 0.
\]

Then, the QP problem of the equivalence Theorem 2 is deduced.

### B Min-max problems

A rational optimality criterion is such that the minimizer evaluates his optimal decision against the worst decision, that the maximizer may choose [12]. In a min-max problem, a function to be maximized w.r.t. the maximizer variables is minimized w.r.t. the minimizer variables. A min-max problem with unseparate constraint is defined by

\[
\min_{x,\sigma} \max_{y \in Y} f(x, y) \\
\text{subject to} \\
G(x, y) \leq 0, \\
x \in X = \{x | g(x) \leq 0\}.
\]

If the constraints are separated, \( G(x, y) \leq 0 \) does not exist. Let \( \sigma \) be an upper bound of \( f \) w.r.t. \( y \) such that \( \max_{x \in X} f(x, y) \leq \sigma \), the min-max problem is transformed into the optimization problem with an infinite number of constraints

\[
\min_{x,\sigma} \max_{y \in Y} f(x, y) \\
\text{subject to} \\
f(x, y) \leq \sigma, \text{ for all } y \in Y, \\
x \in X = \{x | g(x) \leq 0\}.
\]

A method to find a solution to this problem is to solve a series of relaxed problems [12]

\[
\min_{x,\sigma} \max_{y \in Y} f(x, y) \\
\text{subject to} \\
f(x, y^i) \leq \sigma, \text{ for all } y^i \in Y, \ i \in \mathbb{N}_k, \\
x \in X = \{x | g(x) \leq 0\}.
\]

### C Fuzzy quadratic programming

The symmetric approach by Zimmermann [18] may be used for solving fuzzy programming problems. For this approach, membership functions are defined, by using a given aspiration level of the decision maker (DM) for the objective and accepted tolerances for the objective and the constraint functions. An equivalent crisp QP problem is obtained with a quadratic constraint. This particular QP problem can be solved by using van de Panne’s two-phase method [14]

\[12\text{Shimizu and Aiyoshi [12] show that the min-max problem subject to unseparate constraints can be equivalently transformed into a problem subject to separate constraints, by means of the duality theory for nonlinear programming with adequate assumptions.}\]
C.1 Fuzzy QP problem

The fuzzy QP problem may be defined by a convex quadratic objective function together with a bounded feasible region such as [1]

\[
\min_x c'x + \frac{1}{2}x'Qx
\]

subject to

\[
A_i x \leq b_i, \quad i \in \mathbb{N}_m
\]

\[
x \geq 0,
\]

where \(c, x \in \mathbb{R}^n\), \(b \in \mathbb{R}^m\), \(A \in \mathbb{R}^{m \times n}\), \(Q \in \mathbb{R}^{n \times n}\). The vector \(A_i\) denotes the \(i\)th row of matrix \(A\). The symmetric matrix \(Q\) is supposed to be positive semi definite.

C.2 Symmetric fuzzy QP problem

According to Zimmermann [18, 19], the symmetric version of the fuzzy QP problem is

Find \(x\)

such that

\[
c'x + \frac{1}{2}x'Qx \geq z_0,
\]

\[
A_i x \leq b_i, \quad i \in \mathbb{N}_m
\]

\[
x \geq 0,
\]

where \(z_0 \in \mathbb{R}\) is the aspiration level of the DM and \(p_0, p_i, i \in \mathbb{N}_m\) the tolerances for the objective and the set constraints, respectively. The membership function for the objective is defined by

\[
\mu_0(z) = \begin{cases} 
1, & z < z_0, \\
\frac{z_0 + p_0 - z}{p_0}, & z_0 \leq z \leq z_0 + p_0 \\
0, & z \geq z_0 + p_0,
\end{cases}
\]

The membership function for the \(i\)th \((i \in \mathbb{N}_m)\) constraint is also defined by

\[
\mu_i(A_i x) = \begin{cases} 
1, & A_i x < b_i, \\
\frac{b_i + p_i - A_i x}{p_i}, & b_i \leq A_i x \leq b_i + p_i \\
0, & A_i x \geq b_i + p_i.
\end{cases}
\]

An optimal solution is obtained by solving the crisp equivalent QP problem

Find \(\alpha\)

such that

\[
c'x + \frac{1}{2}x'Qx + \alpha p_0 \leq z_0 + p_0,
\]

\[
A_i x + \alpha p_i \leq b_i + p_i, \quad i \in \mathbb{N}_m
\]

\[
\alpha \leq 1,
\]

\[
\alpha, x \geq 0.
\]

C.3 Multiplier method

Let a nonlinear programming problem be defined as [15]

\[
\min_x f(x) \text{ subject to } g(x) = 0 \text{ and } h(x) \leq 0,
\]

where \(g, h\) are nonlinear vectorial functions and \(x\) a vector of variables. The multiplier method is based on the Uzawa algorithm, which is a dual gradient ascent algorithm. The principle of the method may be described by the three steps: i) predict the multipliers \(p^{(k)}\) and \(q^{(k)}\) that are associated with the constraints \(g(x) = 0\) and \(h(x) \leq 0\), ii) then, minimize \(f(x) + p^{(k)}g(x) + q^{(k)}h(x)\), iii) then, update until to convergence as \(p^{(k+1)} = p^{(k)} + c_1 g(x^{(k)})\) and \(q^{(k+1)} = q^{(k)} + c_2 \max\{0, h(x^{(k)})\}\), where the numbers \(c_i, i = 1, 2\) are positive.

C.4 Numerical example

The following numerical example is taken from Bector and Chandra [1], p.77. The fuzzy symmetric QP problem is

Find \((x_1, x_2)\)

such that

\[
2x_1 + x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 \leq 51.88,
\]

\[
4x_1 + 5x_2 \geq 20,
\]

\[
5x_1 + 4x_2 \geq 20,
\]

\[
x_1 + x_2 \leq 30,
\]

\[
x_1, x_2 \geq 0.
\]

Let the tolerances be \(p_0 = 2.12, p_1 = 2, p_2 = 1, p_3 = 3\), the equivalent crisp QP problem is

Find \((x_1, x_2)\)

such that

\[
2x_1 + x_2 + 4x_1^2 + 4x_1x_2 + 2x_2^2 + 2.12\alpha \leq 54,
\]

\[
4x_1 + 5x_2 - 2\alpha \geq 18,
\]

\[
5x_1 + 4x_2 - \alpha \geq 19,
\]

\[
x_1 + x_2 + 3\alpha \leq 33,
\]

\[
\alpha \leq 1,
\]

\[
x_1, x_2, \alpha \geq 0.
\]

The multiplier method package of the software Mathematica [15] uses the primitive \texttt{MultiplierMethod}[f, g, h, x, x0, DualParameter \to True]. This primitive is finding a local solution to a minimization problem where \(f\) is the criterion to be minimized, \(g\) a list (possibly empty) of equality constraints, \(h\) a list (possibly empty) of inequality constraints of the form \(h(x) \leq 0\), \(x\) the list of variables and \(x0\) the initial conditions for \(x\). It returns a list of results \(\{f^*, \{x_1 \to x_1^*, \ldots\}\}\).
The optimum solution of the QP problem, given by the multiplier method is \( x_1^* = .9918, \quad x_2^* = 3.7253, \quad \alpha^* = .8599 \). This result tells that the solution is given by \( x_1^* = .9918 \) and \( x_2^* = 3.7253 \) with a satisfaction level of 86 per cent.

References:


