The computation of the wave motion along the infinite layers with different cross-sections modeled by FE

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Abstract: - Propagation of elastic waves from thicker to a shallower infinite layer is an important phenomenon in engineering, for instance in the earthquake engineering, as the variation of the dimension of the layer has considerable impact on the intensity and the distribution of the wave motion. The paper presents the computing procedure for wave motion field in both layers when the motion is excited by a given displacements in the thicker layer. Modeling of the layers is by FE and the computation is in the frequency domain. The radiation conditions are considered exactly by considering all propagating wave modes in the layer and so the procedure yields exact results within the accuracy of the FE modeling. Theoretical basis of the analysis is briefly presented. The presented numerical examples are for the case of 2D anti-plane shear wave motion of some simple cases, although the presented procedure is applicable to non-homogeneous layers as well. The results of the analysis are briefly discussed.

Key-Words: - Infinite waveguide, Variable cross-section, Wave-modes, Fictitious boundary, FEM, Seismic waves

1 Introduction
The analysis of propagation of waves from thicker to shallower layers is important in several fields of engineering, e.g. seismic excitation of structures in the field of civil engineering, because it may differ essentially from wave motion field in uniform waveguides. When neglecting the dissipation of waves due to the material damping, and not considering the standing waves which occur at the frequencies below the cut-off frequency, the waves in layers propagate theoretically to infinity without attenuation [1,2].

In cases where the height of the layer at some point shrinks the propagating waves are partially reflected, rendering a combination of standing waves and propagating waves. Only some waves enter the second layer. There these waves, with the wave length smaller than the cut-off wave length, propagate without attenuation in wave modes. Other waves are standing which attenuate rapidly. The essential feature in the analysis of this problem is that we have to consider the radiation conditions which render correct solution of wave equation for propagating waves towards infinity.

The analytical solution is practically hardly feasible even for most simplified cases thus a variety of numerical methods has been developed for this purpose. If we glance over them, we could describe them roughly as follows. Boundary element methods satisfy radiation conditions, but are not simple to apply for complex cases, see for example [3,4]. By finite-element methods the radiation conditions are satisfied only by using special elements on the fictitious boundary, or by evaluating certain computational phases analytically, see for instance [5]. Operator methods require the implementation of special operators on the fictive boundary [6]. A special approach using an original definition of Sommerfeld conditions is presented by [7]. More sophisticated approach for wave guides with the variable cross-section is presented by [8]. A more complete treatment of these problems is presented in [9]. We can classify the available methods superficially and briefly as being either a great deal sophisticated and in certain cases exact, or simple and considerably approximate. Unfortunately, a simple and exact method, which in addition is simple to apply and understand by practical engineers, is not available.

The author of this paper, with co-authors, has presented methods for the analysis of waves in homogeneous in inhomogeneous waveguides, as well as for the layered space over rigid and elastic half space [10,11]. Unfortunately these approaches, based on the dynamic flexibility matrix, can not be
applied directly to the case considered here because of the numerical instability of the procedure. The procedure is thus appropriately altered and is presented in the next chapter.

2 The Outline of the Computing Procedure and the Key Formulas

With the regard to the Fig.1, the boundary conditions for the domain of the considered part of the Layer 1 are all known in advance except on the fictitious boundary 1. Thus, on the boundary 0 we have known excitation displacements, on the clamped boundary the displacements are zero, and in on the free surface the tractions are zero. On the fictitious boundary 1 the displacements and stresses are unknown and are equal to an unknown combination of wave modes which travel and/or decay in the layer 2 in the positive direction of x axis towards infinity. They satisfy radiation conditions and are presented by Equation 1, where \( \Psi_{n,U} \) are displacements wave modes and \( a_n \) their amplitudes (the weighting factors), \( N \) is the number of FE mesh nodes on the fictitious boundary 1.

\[
\sum_{n=1}^{N} a_n \Psi_{n,U} (y), e^{jz_n x} = 0 \quad (1)
\]

The procedure and the theory for the computation of wave nodes is known, see for instance [10,11]. We are presenting it here only briefly. In the FE discretized model the wave modes \( \Psi_n \) are the solution of the eigenvalue problem of the transfer matrix \( T_{cell} \) of the “cell” in the Layer 2, Equation 2.

\[
\begin{bmatrix}
[T_{cell} - \lambda_n I] \\
\Psi_{n,U}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix} \quad (2)
\]

\( \Psi_{n,U} \) stands for the n-th displacements modes, \( \Psi_{n,F} \) for the belonging mode of nodal forces. The meaning of eigenvalue \( \lambda \) is clear from Equation 3.

\[
\lambda_n = \exp(z_n, d) = \exp\left(-\frac{i \omega d}{c_n}\right) \quad (3)
\]

In the Equation 3, “d” stands for the width of the “cell”, \( c_n \) for phase velocity of the n-th wavemode while the meaning of the exponential factor “z” is straightforward.

It is important to note that the eigenvectors of the nodal forces on the fictitious boundary 1 have the same weighting factors as the eigenvectors of the nodal displacements, Equation 4.

\[
u(x, y) = \sum_{n=1}^{N} a_n \Psi_{n,F} (y), e^{jz_n x} \quad (4)
\]

After forming FE mesh of the considered part of the Layer 1 and computing dynamic stiffness matrix \( K \), we set the governing equation, Equation 5, for the displacements and the forces field. The dynamic stiffness matrix, displacement and forces vectors are partitioned according to the values which belong to mesh nodes on the excitation boundary 0 (index 0), internal nodes (index C) and nodes on the fictitious boundary 1 (index 1).

\[
\begin{bmatrix}
K_{00} & K_{0C} & K_{01} \\
K_{C0} & K_{CC} & K_{C1} \\
K_{10} & K_{IC} & K_{11}
\end{bmatrix}
\begin{bmatrix}
U_0 \\
U_C \\
U_1
\end{bmatrix} = \begin{bmatrix}
P_0 \\
P_C \\
P_1
\end{bmatrix} \quad (5)
\]

From this equation we have to express displacements \( U_0 \) by displacements and forces on the fictitious boundary 1, Equation 6.

\[
(K_{10} - K_{IC} K_{CC}^{-1} K_{CO}) U_0 = (K_{IC} K_{CC}^{-1} K_{CI} - K_{11}) U_1 + P_1 \quad (6)
\]

Substituting Equations 1 and 4 into Equation 6 we get Equation 7,

\[
(K_{10} - K_{IC} K_{CC}^{-1} K_{CO}) \Psi_{U} = [(K_{IC} K_{CC}^{-1} K_{CI} - K_{11}) \Psi_{U} + \{a\} \{a\} \quad (7)
\]

where \( \Psi_{U} \) and \( \Psi_{U} \) are the matrices of all wavemodes for displacements and forces, respectively, propagating in x direction, and \( \{a\} \) is vector of weighting factors. It is worth noting that, regarding Equations 1 in 4, we consider that in the above equation x coordinate is zero 0 which yields values on the fictitious boundary.

First we compute the left hand side of the above equation and then we solve it on the weighting factors \( \{a\} \). Then according to the Equation 1 all boundary displacements are known and the computation of displacement field in Layer 1 is computed in the well known way. Finally, the displacements in Layer 2 are computed with the aid of formula 1.
3 Numerical Examples

The presented examples are two dimensional cases of anti-plane shear wave motion analyzed in the frequency domain. The wave motion is governed by the Equation:

\[ \nabla^2 u + k^2 u = 0. \]  

The disposition of the layers is presented symbolically in Fig.1. The Height of the first layer is 10 meters, its considered length is 20 m, and the height of second layer is 5 meters. The shear wave velocity, the shear module and the density are all equal a unit. The excitation is by displacement at the boundary which varies along the y axis according to the sinus function from zero \( \pi /2 \). The shape of this excitation is actually equal to the first wave mode in the Layer 1. The nodes of computational FE mesh are 0.5 meter apart.

The wavemode shapes in layer 2 are presented in Fig.2 and do not depend on the frequency. Two cases of frequency excitation are considered. The first one is 1 radian per second, which renders only standing wavemodes which are decaying in the Layer 2. Table 1 presents eigenvalues \( \lambda \) (Equation 2), which represent the decaying factor of waves per meter. However, the first two values have absolute value 1 meaning that these wavemodes are traveling without decay, while the remaining ones are standing rapidly decaying wavemodes. In the Table is presented also exponential factor \( z \) (Equation 3), which yields the velocity of wavemodes. Weighting factors “a” demonstrate that the displacements on the fictitious boundary 1 consists prevailingly of the first two modes. The second example considers the frequency of excitation 0.3 rad/s. In this case all wavemodes are standing; see \( \lambda \) in Table 1, while the displacements field in Layer 1 consists practically of only the first mode. The absolute and real part of displacements field for both cases are presented in Figures 3 to 6. The impact of decaying and propagating waves on the displacement field in Layer 2 can be easily observed.

4 Conclusion

As there are no approximations of the computed wave field in the presented procedure, except for the FE modeling, it yields accurate results because the radiation conditions are satisfied exactly. Two advantages are worth pointing out: the computing procedure is simple and employable by using standard computer routines. The procedure also yields good insight of the contents of wavemodes in the displacement field, their velocity and possible decay. Finally, the procedure can well be applied to more complex cases, e.g. inhomogeneous layered waveguides.

References:
Figure 1. Symbolic presentation of layers, fictitious boundary and the excitation by displacements.

Figure 2. First five wavemodes normalized to yield a unit displacement on the surface.

<table>
<thead>
<tr>
<th>Consecutive number of mode</th>
<th>Eigenvalue $\lambda$</th>
<th>Exponential factor $z$</th>
<th>Absolute weighting faktor</th>
<th>Eigenvalue $\lambda$</th>
<th>Exponential factor $z$</th>
<th>Absolute weighting faktor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5841-0.8117i</td>
<td>-0.9470i</td>
<td>1.2162</td>
<td>0.9097</td>
<td>-0.0946</td>
<td>1.9948</td>
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<tr>
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<td>-0.3020i</td>
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<td>0.4038</td>
<td>-0.9068</td>
<td>0.1423</td>
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<td>0.1192</td>
<td>0.2011</td>
<td>-1.6037</td>
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<td>-6.9918</td>
<td>0.0017</td>
</tr>
</tbody>
</table>

Table 1. Eigenvalues $\lambda$ (Equation 3), exponential factors $z$ (Equations 1,3), and weighting factors (Eqs 1,4,7) of wavemodes for the frequency of excitation 1 and 0.3 radians per second.
Figure 3. Absolute displacements for the excitation with the frequency 1.0 rad/sec.

Figure 4. Absolute displacements for the excitation with the frequency 0.3 rad/sec.
Figure 5. Real part of displacements at an instant for the excitation with the frequency 1.0 rad/sec.

Figure 6. Real part of displacements at an instant for the excitation with the frequency 0.3 rad/sec.