Design of linear and quadratic filtering algorithms using uncertain observations from multiple sensors with correlated uncertainty

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Abstract: In this paper, filtering algorithms are derived for the least-squares linear and quadratic estimation problems in linear systems with uncertain observations coming from multiple sensors with different uncertainty characteristics. It is assumed that, at each sensor, the state is measured in the presence of additive white noise and that the Bernoulli random variables describing the uncertainty are correlated at consecutive sampling times but independent otherwise. The least-squares linear estimation problem is solved by using an innovation approach, and the quadratic estimation problem is reduced to a linear estimation one in a suitable augmented system. The performance of the linear and quadratic estimators is illustrated by a numerical simulation example wherein a scalar signal is estimated from correlated uncertain observations coming from two sensors with different uncertainty characteristics.

Key–Words: Linear and quadratic estimation, uncertain observations, multiple sensors

1 Introduction

Recently, many papers concerning the state estimation problem from measurements coming from different multiple sensors are emerging in several application fields, such as network communication systems incorporating heterogeneous measurement devices (see e.g. [4], [6] and [8], among others).

In classical estimation theory, the observation is always assumed to contain the state to be estimated. However, there exist a broad class of real-world problems in which the state appears in the observation in a random manner, such as problems where there are intermittent failures in the observation mechanism, fading phenomena in propagation channels, target tracking, accidental loss of some measurements or data inaccessibility during certain times. These situations are characterized by the fact that the state is not always present in the observations but there is a nonzero probability that the measurement contains only noise (false alarm probability). This consideration is modelled including in the observation equation not only an additive noise, but also a multiplicative noise consisting of a sequence of Bernoulli random variables taking the value one if the observation contains state plus noise, or the value zero if it is only noise (uncertain observations).

Due to this multiplicative noise, systems with uncertain observations are not Gaussian in general and the practical computation of the optimal least-squares estimator is not a simple task, as occurs in other considerable number of situations in which the widely used assumption of Gaussian additive noises must be removed in order to obtain a more realistic statistical model. This difficulty has motivated the necessity of looking for suboptimal estimators which are easier to obtain, such as linear or, more generally, polynomial estimators which improve the extensively used linear ones.

The least-squares linear and polynomial estimation problem in this kind of systems has been studied by several authors under different approaches and hypotheses on the processes involved (see [1], [2], [10], among others).

In the above papers, the variables modelling the uncertainty in the observations are assumed to be independent, so the distribution of the multiplicative noise is fully determined by the probability that each particular observation contains the state to be estimated. More general situations, in which such independence assumption is not valid since the variables modelling the uncertainty are correlated at consecutive instants, have been considered in [5], using a state-space approach, and in [9] under a covariance approach.

On the other hand, also the least-squares linear estimation problem in systems with uncertain observa-
tions transmitted by multiple sensors whose statistical properties are assumed not to be the same for all the sensors has been recently studied by several authors under different approaches and hypotheses on the processes (see e.g. [3] using a state-space approach, [7] under a covariance approach, and references therein).

In this paper recursive algorithms for the least-squares linear and quadratic filtering problems from correlated uncertain observations coming from multiple sensors with different uncertainty characteristics are proposed. To approach the quadratic estimation problem we use the technique proposed in [11], which consists of augmenting the state and observation vectors, by aggregating their second-order powers defined by the Kronecker product, thus obtaining a new augmented system and reducing the quadratic estimation problem from the original system to the linear estimation problem from the augmented system. Recursive linear filtering algorithms are derived by using an innovation approach.

The performance of the proposed filtering algorithm is illustrated by a numerical simulation example where the state of a first-order autoregressive model is estimated from uncertain observations coming from two sensors with different uncertainty characteristics and correlated at consecutive sampling times.

2 Hypotheses on the model

Consider the following linear stochastic system with correlated uncertain observations coming from multiple sensors:

\[ x_k = F_{k-1}x_{k-1} + w_{k-1}, \quad k \geq 1 \]

\[ y^i_k = \gamma^i_k H_k^i x_k + v^i_k, \quad k \geq 1, \quad i = 1, \ldots, m \]

where \( \{x_k; k \geq 0\} \) is the \( n \)-dimensional state process and \( \{y^i_k; k \geq 1\} \) denotes the scalar observation process from the \( i \)-th sensor. The additive noises \( \{w_k; k \geq 0\} \) and \( \{v^i_k; k \geq 1\} \) are white processes, and the multiplicative noises \( \{\gamma^i_k; k \geq 1\} \) are sequences of Bernoulli random variables representing the presence or absence of the state \( x_k \) in the observations \( y^i_k \); more specifically, if \( \gamma^i_k = 1 \) the state is present in the observation coming from the \( i \)-th sensor at time \( k \), while if \( \gamma^i_k = 0 \) such observation contains only noise, \( v^i_k \). It is also assumed that, at each sensor, the uncertainty in the observation at time \( k \) depends on the uncertainty at time \( k - 1 \), but it is independent of uncertainties at times previous to \( k - 1 \); this is formulated by imposing the stochastic independence of the Bernoulli variables \( \gamma^i_k \) and \( \gamma^s_k \) when \( |k-s| \geq 2 \). This special form of correlation covers those models in which the state cannot be missing in two successive observations; for example, signal transmission problems where any failure in a sensor (that is, when there is no signal to transmit) is immediately detected at the transmitting end and the failed sensor is replaced.

To simplify the notation, the observation equation is rewritten in a compact form as follows:

\[ y_k = \Upsilon_k H_k x_k + v_k, \quad k \geq 1, \] (1)

where

\[ y_k = (y^1_k, \ldots, y^m_k)^T, \quad \Upsilon_k = \text{Diag}(\gamma^1_k, \ldots, \gamma^m_k), \]

\[ H_k = (H^1_k, \ldots, H^m_k)^T, \quad v_k = (v^1_k, \ldots, v^m_k)^T. \]

In this paper, the least-squares (LS) linear and quadratic filtering problems of the state \( x_k \) based on the observations \( y_1, \ldots, y_k \) defined by (1) are addressed.

2.1 Hypotheses for LS linear estimation

It is known that if the state \( x_k \) and the observations \( y_1, \ldots, y_k \) have finite second-order moments, the LS linear filter of \( x_k \) is the orthogonal projection onto the space of \( n \)-dimensional random variables obtained as linear transformations of the observations \( y_1, \ldots, y_k \). The following hypotheses specify the first- and second-order moments required in the study of this problem, as well as the statistical properties assumed about the initial state and noise processes:

- The initial state \( x_0 \) is a random vector with \( \mu_0 = E[x_0] \) and \( P_0 = \text{Cov}[x_0] \).

- The state noise \( \{w_k; k \geq 0\} \) is a zero-mean white sequence with \( Q_k = \text{Cov}[w_k], \forall k \geq 0 \).

- For \( i = 1, \ldots, m \), the noise \( \{\gamma^i_k; k \geq 1\} \) is a sequence of independent Bernoulli random variables with \( P[\gamma^i_k = 1] = p^i_k \). The variables \( \gamma^i_k \) and \( \gamma^s_k \) are assumed to be independent if \( |k-s| \geq 2 \), and \( \text{Cov}[\gamma^i_{k+1}, \gamma^j_k] \) are known for \( k \geq 1, i = 1, \ldots, m \).

- For \( i = 1, \ldots, m \), the sensor additive noises, \( \{v^i_k; k \geq 1\} \), are zero-mean white processes with known variances.

- The initial state \( x_0 \) and the noise processes, \( \{w_k; k \geq 0\}, \{\gamma^i_k; k \geq 1\} \) and \( \{v^i_k; k \geq 1\} \), for \( i = 1, \ldots, m \), are mutually independent.

2.2 Hypotheses for LS quadratic estimation

The LS quadratic estimator of \( x_k \) based on the observations \( y_1, \ldots, y_k \) is the orthogonal projection of
\( x_k \) onto the space of \( n \)-dimensional linear transformations of \( y_1, \ldots, y_k \) and their second-order powers, \( y_1^2, \ldots, y_k^2 \), defined by the Kronecker product, \( y_k^2 = y_k \otimes y_k \). Then, in order to analyze the LS quadratic estimation problem, the existence of the second-order moments of the vectors \( y_k^2 \) is required and the following additional hypotheses are assumed:

- The initial state \( x_0 \) is a fourth-order random vector with \( P_0^{(3)} = \text{Cov}[x_0, x_0^2] \) and \( P_0^{(4)} = \text{Cov}[x_0^2, x_0^2] \).
- The state noise \( \{w_k; k \geq 0\} \) is a fourth-order process with \( Q_k^{(3)} = \text{Cov}[w_k, w_k^2] \) and \( Q_k^{(4)} = \text{Cov}[w_k^2, w_k^2] \), \( \forall k \geq 0 \).
- The sensors additive noise, \( \{v_k; k \geq 1\} \), is a fourth-order process with \( H_k^{(3)} = \text{Cov}[v_k, v_k^2] \) and \( R_k^{(4)} = \text{Cov}[v_k^2, v_k^2] \).

### 3 LS linear estimation problem

In this section, the LS linear estimation problem of the state \( x_k \) based on the available observations up to time \( k \), \( y_1, \ldots, y_k \), is addressed under the hypotheses established in Section 2.1. Next, the properties of the noises in (1) are specified.

- \( \{v_k; k \geq 1\} \) is a zero-mean white sequence with covariances \( R_k \), defined from the variances of their components \( v_k^t \).

- The random matrices \( \Upsilon_k \) satisfy

\[
\Upsilon_k G_{m \times m} \Upsilon_s = (C_{\Upsilon_k} G_{T \Upsilon_s}) \circ G_{m \times m}
\]

for any matrix \( G_{m \times m} \), where \( C_{\Upsilon_k} = (\gamma_{1k}^1, \ldots, \gamma_{mk}^m)^T \) and \( \circ \) denotes the Hadamard product. The random vectors \( C_{\Upsilon_k} \) and \( C_{\Upsilon_s} \) are independent for \( |k - s| \geq 2 \) and the covariance matrices of \( C_{\Upsilon_k} \) and \( C_{\Upsilon_s} \) for \( s = k, k - 1 \) are given by

\[
K_{k,s}^0 = \text{Diag} \left( \text{Cov}[\gamma_{11}^1, \gamma_{11}^1], \ldots, \text{Cov}[\gamma_{1m}^m, \gamma_{1m}^m] \right).
\]

- \( x_0, \{w_k; k \geq 1\}, \{v_k; k \geq 1\} \) and \( \{\Upsilon_k; k \geq 1\} \) are mutually independent.

Our aim is to obtain the filter, \( \hat{x}_{k/k} \), from a recursive algorithm and, for this purpose, an innovation approach will be used. Since the LS linear estimator based on the observations is equal to that based on the innovations, to address the estimation problem the observation process will be replaced by the innovation one. The innovation at time \( k \) is defined as \( \nu_k = y_k - \hat{y}_{k/k-1} \) where \( \hat{y}_{k/k-1} \), the one-stage linear predictor of \( y_k \), is the orthogonal projection of \( y_k \) onto the \( m \)-dimensional linear space generated by \( v_1, \ldots, v_{k-1} \). Since the innovations constitute a white process, this methodology allows us to find the projection by separately projecting onto each of the previous orthogonal vectors; that is,

\[
\hat{y}_{k/k-1} = \sum_{j=0}^{k-1} E[y_k v_j^T] \Pi_j^{-1} \nu_j, \quad k \geq 2; \quad \hat{y}_{1/0} = E[y_1],
\]

being \( \Pi_j = E[v_j v_j^T] \) the covariance of \( \nu_j \).

In a similar way, the replacement of the observation process by the innovation one leads to

\[
\hat{x}_{k/k} = \sum_{j=1}^{k} E[x_k v_j^T] \Pi_j^{-1} \nu_j, \quad k \geq 1,
\]

and denoting \( G_k = E[x_k v_k^T] \), the following expression for the filter \( \hat{x}_{k/k} \) in terms of the predictor, \( \hat{x}_{k/k-1} \), is obvious:

\[
\hat{x}_{k/k} = \hat{x}_{k/k-1} + G_k \Pi_k^{-1} \nu_k, \quad k \geq 1; \quad \hat{x}_{0/0} = \mu_0.
\]

Next, we obtain the state predictor \( \hat{x}_{k/k-1} \), the innovation \( \nu_k \) and its covariance matrix \( \Pi_k \), and the gain matrix \( G_k \), which jointly with (3) constitute the proposed recursive linear filtering algorithm.

### 3.1 Linear filtering algorithm

**State predictor.** From the orthogonal projection lemma, it is immediate that

\[
\hat{x}_{k/k-1} = F_{k-1} \hat{x}_{k-1/k-1}, \quad k \geq 1.
\]

**Innovation process.** When, for each \( i = 1, \ldots, m \), the Bernoulli variables \( \{\gamma_i^k; k \geq 1\} \) modelling the uncertainty at the \( i \)-th sensor are independent, all the information prior to time \( k \) required to estimate \( y_k^i \) is provided by the one-stage predictor \( \hat{x}_{k/k-1} \); specifically, \( \nu_{k/k-1}^i = p_k H_k^i \hat{x}_{k/k-1} \). However, for the problem at hand, the correlation between \( \gamma_k^1 \) and \( \gamma_k^2 \), which must be considered for such estimation, is not contained in \( \hat{x}_{k/k-1} \). Therefore, to obtain the current innovation \( \nu_k \), it is necessary to find the new expression for the one-stage predictors of \( y_k^i \), which provide the one-stage predictor (2) Taking into account the model hypotheses,

\[
E[y_k v_j^T] = \Upsilon_k^p H_k E[x_k v_j^T], \quad i \leq k - 2; \quad E[y_k v_k^T] = E[C_{\Upsilon_k}^T C_{\Upsilon_k}] \circ (H_k F_{k-1} \Pi_{k-1} H_k^T) - \Upsilon_k^p H_k E[x_k y_{k-1/k-2}^T], \quad k \geq 2,
\]
where $Y_k^p = E[Y_k]$ and $s_k = E[x_k x_k^T]$. Substituting these expectations in (2), we obtain

$$
\hat{y}_{k|k-1} = \mathcal{Y}_k^p H_k \sum_{i=1}^{k-1} E[x_k \nu_{i}^T] \Pi_{k-1}^{-1} \nu_i + \left( E[C \gamma_k \mathcal{Y}_{k-1}^p] \right)
\circ (H_k F_{k-1} s_{k-1} H_k^T) \Pi_{k-1}^{-1} \nu_{k-1} - \mathcal{Y}_k^p H_k E[x_k y_{k-1}^T] \Pi_{k-1}^{-1} \nu_{k-1}, \quad k \geq 2.
$$

Since $E[x_k y_{k-1}^T] = F_{k-1} s_{k-1} H_k^T \mathcal{Y}_k^p - 1$, we have

$$
\mathcal{Y}_k^p H_k E[x_k y_{k-1}^T] = (E[C \gamma_k \mathcal{Y}_{k-1}^p]) \circ (H_k F_{k-1} s_{k-1} H_k^T)
$$

and from $E[C \gamma_k \mathcal{Y}_{k-1}^p] = E[C \gamma_k \gamma_{k-1}^p] = K_{k,k-1}^\gamma$, it is concluded that

$$
\hat{y}_{k|k-1} = \mathcal{Y}_k^p H_k \hat{x}_{k|k-1} + L_{k-1} \Pi_{k-1}^{-1} \nu_{k-1}, \quad k \geq 2;
\hat{y}_{1|0} = \mathcal{Y}_1^p H_1 \hat{x}_{1|0}.
$$

(5)

Where

$$
L_{k} = K_{k+1,k}^\gamma \circ (H_{k+1} F_k s_k H_k^T), \quad k \geq 1,
$$

and $s_k$ is recursively obtained by

$$
s_k = F_{k-1} s_{k-1} F_{k-1}^T + Q_{k-1}, \quad k \geq 1; \quad s_0 = P_0 + \mu_0 \nu_0^T.
$$

Hence, $\nu_k$ is obtained as a linear combination of the old observation, the state predictor and the previous innovation:

$$
\nu_k = y_k - \mathcal{Y}_k^p H_k \hat{x}_{k|k-1} - L_{k-1} \Pi_{k-1}^{-1} \nu_{k-1}, \quad k \geq 2;
\nu_1 = y_1 - \mathcal{Y}_1^p H_1 \hat{x}_{1|0}.
$$

(6)

From (6) and the hypotheses on the model, the following expression for $\Pi_k$, the covariance matrix of the innovation $\nu_k$, is obtained after some manipulations:

$$
\Pi_k = (\mathcal{Y}_k^p (I - \mathcal{Y}_k^p)) \circ (H_k s_k H_k^T) + R_k
+ \mathcal{Y}_k^p H_k P_{k,k-1} F_{k-1}^T \mathcal{Y}_k^p - L_{k-1} \Pi_{k-1}^{-1} L_{k-1}^T
-L_{k-1} \Pi_{k-1}^{-1} G_{k-1}^T F_{k-1}^T H_k^T \mathcal{Y}_k^p
- \mathcal{Y}_k^p H_k F_{k-1} G_{k-1} \Pi_{k-1}^{-1} L_{k-1}^T, \quad k \geq 2;
$$

$$
\Pi_1 = (\mathcal{Y}_1^p (I - \mathcal{Y}_1^p)) \circ (H_1 s_1 H_1^T) + R_1
+ \mathcal{Y}_1^p H_1 P_{1,0} H_1^T \mathcal{Y}_1^p,
$$

(7)

where $I$ denotes the identity matrix of appropriate dimensions and $P_{k,k-1}$, the prediction error covariance matrix, verifies

$$
P_{k,k-1} = F_{k-1} P_{k-1,k-1} F_{k-1}^T + Q_{k-1}, \quad k \geq 1; \quad P_{0,0} = P_0
$$

(8)

with $P_{k,k}$, the filtering error covariance matrix, satisfying

$$
P_{k,k} = P_{k,k-1} - G_k \Pi_{k-1} G_k^T, \quad k \geq 1.
$$

(9)

**Gain matrix.** Using again (5), the following expression for $G_k = E[x_k y_k^T] - E[x_k y_{k|k-1}^T]$ is obtained:

$$
G_k = P_{k,k-1} H_k^T \mathcal{Y}_k^p - F_{k-1} G_{k-1} \Pi_{k-1} L_{k-1}^T, \quad k \geq 2;
\Gamma_1 = P_{1,0} H_1^T \mathcal{Y}_1^p.
$$

(10)

The proposed linear filtering algorithm is constituted by equations (3), (4), (6)-(10).

### 4 LS quadratic estimation problem

Now a recursive algorithm to obtain the quadratic filter, $\hat{x}_{k,k}^2$, of the state $x_k$ from the observations $y_1, \ldots, y_k$ defined in (1) is derived under the hypotheses specified in sections 2.1 and 2.2. To obtain this estimator, consider the following augmented state and observation vectors, which are defined by aggregating the original vectors with their second-order Kronecker powers:

$$
X_k = \begin{pmatrix} x_k & x_k^2 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} y_k & y_k^2 \end{pmatrix}.
$$

Clearly, the space of $n$-dimensional linear transformations of $\gamma_1, \ldots, \gamma_k$ is equal to the space of $n$-dimensional linear transformations of $y_1, \ldots, y_k$ and $y_1^2, \ldots, y_k^2$. Then, the LS quadratic estimator, $\hat{x}_{k,k}^2$, is the LS linear estimator of $x_k$ based on $\gamma_1, \ldots, \gamma_k$, and it is obtained by extracting the first $n$-dimensional block of the LS linear estimator of $X_k$ based on $\gamma_1, \ldots, \gamma_k$. Next, the relation between the augmented vectors $X_k$ and $\gamma_k$, as well as their statistical properties are analyzed.

#### 4.1 Augmented system

For simplicity, the estimation problem will be addressed for the centered augmented vectors, $X_k = X_k - E[X_k]$ and $Y_k = \gamma_k - E[\gamma_k]$. Since the $n$-dimensional linear space generated by $\gamma_1, \ldots, \gamma_k$ is equal to that generated by $y_1, \ldots, y_k$, the required LS linear estimator of $X_k$ based on $\gamma_1, \ldots, \gamma_k$ will be obtained from that of $X_k$ based on $y_1, \ldots, y_k$.

Using the Kronecker product properties and the system hypotheses it is deduced that the centered augmented vectors satisfy the following equations:

$$
X_k = F_k X_{k-1} + W_{k-1}, \quad k \geq 1
Y_k = D_k \mathcal{H}_k X_k + V_k, \quad k \geq 1
$$

(11)

where $F_k = \text{Diag} \left( F_{k}, F_{k}^2 \right)$, $D_k = \text{Diag} \left( \gamma_k, \gamma_k^2 \right)$, $\mathcal{H}_k = \text{Diag} \left( H_k, H_k^2 \right)$.

$$
W_k = \left( I + K \right) ((F_k x_k) \otimes w_k) + w_k^2 - \text{vec}(Q_k)
$$
\[ V_k = \left( I + K \right) \left( T_k H_k x_k \otimes v_k \right) + v_k^{[2]} - vec(R_k) \]

\[ + \left( D_k^T - D_k^T \right) H_k \mathcal{E}[X_k], \]

with \( D_k^T = E \left[ D_k^T \right] \), \( K \) the commutation matrix of appropriate dimensions, and vec the operator that vectorizes a matrix.

In the following lemmas the statistical properties of the processes involved in (11) are established.

**Lemma 1** The noise \( \{ W_k; k \geq 0 \} \) is a sequence of zero-mean uncorrelated random vectors with

\[ E[W_k W_k^T] = \mathcal{Q}_k = \begin{pmatrix} Q_k & Q_k^{12T} \\ Q_k^{12} & Q_k^{22} \end{pmatrix} \]

where

\[ Q_k^{12} = \left( (F_k \mu_k)^T \otimes Q_k \right) \left( I + K \right) + Q_k^{(3)} \]

\[ Q_k^{22} = \left( I + K \right) \left( \left( F_k \mu_k \right)^T \otimes Q_k \right) \left( I + K \right) \]

\[ + \left( (F_k \mu_k)^T \otimes Q_k^{(3)} \right) (I + K) \]

\[ + \left( I + K \right) \left( \left( F_k \mu_k \right) \otimes Q_k^{(4)} \right) + Q_k^{(4)} \]

with \( \mu_k = E[x_k] = F_{k-1} \mu_{k-1}, k \geq 1 \).

**Lemma 2** The noise \( \{ V_k; k \geq 1 \} \) is a sequence of zero-mean random vectors with \( E[V_k V_k^T] = R_{k,s}^V = 0 \) for \( |k - s| \geq 2 \) and

\[ R_k^V = \begin{pmatrix} R_k & R_k^{12} \\ R_k^{12T} & R_k^{22} \end{pmatrix} + \text{Cov}[C_k^{(2)}] \circ \left( H_k E[X_k] E[X_k^T] H_k^T \right) \]

\[ R_{k,s}^V = \text{Cov}[C_k^{(2)}, C_{k-1}^{(2)}] \circ \left( H_k E[X_k] E[X_{k-1}] H_{k-1}^T \right) \]

where

\[ C_k^{(2)} = \begin{pmatrix} C_k^{(2)}_{k,k} & C_k^{(2)}_{k,s} \\ C_k^{(2)*}_{s,k} & C_k^{(2)*}_{s,s} \end{pmatrix}, \]

\[ E[X_k] = \begin{pmatrix} \mu_k \\ vec(s_k) \end{pmatrix}, \]

\[ R_k^{12} = \left( \left( T_k^p H_k \mu_k \right)^T \otimes R_k \right) (I + K) + R_k^{(3)} \]

\[ R_k^{22} = \left( I + K \right) \left( \left( T_k^p H_k \mu_k \right)^T \otimes R_k \right) (I + K) \]

\[ + \left( \left( T_k^p H_k \mu_k \right) \otimes R_k^{(3)} \right) (I + K) \]

\[ + \left( I + K \right) \left( \left( T_k^p H_k \mu_k \right) \otimes R_k^{(4)} \right). \]

**Lemma 3** The initial state \( X_0 \) is a zero-mean random vector with covariance

\[ S_0 = \begin{pmatrix} P_0 & P_0^{(3)} \\ P_0^{(3)*} & P_0^{(4)} \end{pmatrix}. \]

Moreover,

- \( X_0 \), \( \{ W_k; k \geq 0 \} \) and \( \{ V_k; k \geq 1 \} \) are uncorrelated.
- For all \( k \geq 1 \), the matrix \( D_k^T \) is independent of \( (X_0, \{ W_k; k \geq 0 \}, V_1, \ldots, V_{k-2}) \).

To obtain the linear filter \( \hat{X}_{k/k} \), consider the augmented innovations \( I_i = \hat{Y}_i - \hat{Y}_{i-1}, i \leq k \). As in Section 3, we have that

\[ \hat{X}_{k/k} = \hat{X}_{k/k-1} + G_k \Xi_{k-1} I_k, \quad k \geq 1; \quad \hat{X}_{0/0} = 0 \]

where \( G_k = E[X_k I_k^T] \) and \( \Xi_k = E[I_k^T I_k] \).

In view of the properties established in lemmas 1-3 and reasoning as in Section 3.1, we obtain the state predictor \( \hat{X}_{k/k-1} \), the innovation \( I_k \) and its covariance matrix \( \Xi_k \), and the gain matrix \( G_k \), which jointly with (12) constitute the following recursive algorithm for the linear estimators \( \hat{X}_{k/k} \) from which the quadratic estimators \( \hat{X}_{k/k}^q \) are obtained just extracting the first \( n \) entries.

### 4.2 Linear filtering algorithm for the augmented state

The linear predictor and filter are given by

\[ \hat{X}_{k+1/k} = \hat{X}_{k+1/k-1} - L_{k-1} \Xi_{k-1}^{-1} I_k, \quad k \geq 2; \quad \mathcal{I}_1 = \mathcal{I}_1 \]

The innovation satisfies

\[ \mathcal{I}_k = Y_k - D_k^p H_k \hat{X}_{k/k-1} - L_{k-1} \Xi_{k-1} I_k, \quad k \geq 2; \quad \mathcal{I}_1 = \mathcal{I}_1 \]

with

\[ L_k = \text{Cov} \left[ C_{k+1}^{(2)} C_k^{(2)*} \right] \circ \left( H_{k+1} F_k S_k H_k^T \right) + R_{k+1,k}^V. \]

The innovation covariance matrix \( \Xi_k \) is

\[ \Xi_k = \text{Cov} \left[ C_k^{(2)} \right] \circ \left( H_k S_k H_k^T \right) + D_k^p H_k \Xi_{k-1} \Xi_{k-1}^T H_k^T \]

\[ - D_k^p H_k \Xi_{k-1} \Xi_{k-1}^T H_k^T + R_{k,k}^V, \quad k \geq 2; \]

\[ \Xi_1 = \text{Cov} \left[ C_1^{(2)} \right] \circ \left( H_1 S_1 H_1^T \right) + D_1^p H_1 \Xi_{0/0} H_1^T + R_{1,1}^V. \]

where \( S_k \) is recursively calculated from

\[ S_k = F_{k-1} S_{k-1} F_{k-1}^T + \mathcal{Q}_{k-1}, \quad k \geq 1, \]

and the prediction and filtering error covariance matrices are recursively calculated from

\[ \Sigma_k/k-1 = F_{k-1} \Sigma_{k-1/k-1} F_{k-1}^T + \mathcal{Q}_{k-1}, \quad k \geq 1 \]

\[ \Sigma_{k/k} = \Sigma_{k/k-1} - G_k \Pi_k \Xi_k^{-1} G_k^T, \quad k \geq 1; \quad \Sigma_0/0 = S_0. \]
Finally, the gain matrix is given by
\[ G_k = \Sigma_{k/|k-1} H_k^T D_k^P - F_{k-1} G_{k-1} \Xi_{k-1} L_{k-1}^T, \quad k \geq 2; \]
\[ G_1 = \Sigma_{1/0} H_1^T D_1^P. \]

5 Numerical simulation example

To show the effectiveness of the proposed estimators, we ran a program in MATLAB, simulating at each iteration the state and the observed values and providing the linear and quadratic filtering estimates, as well as the corresponding error covariance matrices.

Consider a scalar first-order autoregressive model,
\[ x_k = 0.95x_{k-1} + w_{k-1}, \quad k \geq 1 \]
where the initial state is a zero-mean Gaussian variable with \( \text{Var}[x_0] = 1 \) and \( \{w_k; k \geq 0\} \) is a zero-mean white Gaussian noise with \( \text{Var}[w_k] = 0.1 \).

Consider two sensors whose uncertain measurements, \( y_i^k = \gamma_i^k x_k + v_i^k, \quad k \geq 1, \quad i = 1, 2 \), are perturbed by independent additive zero-mean white noises, \( \{v_i^k; k \geq 1\}, \quad i = 1, 2 \), with the following probability distributions
\[ P[v_i^1 = -8] = \frac{1}{8}, \quad P[v_i^1 = 8] = \frac{7}{8}, \quad \forall k \geq 1; \]
\[ P[v_i^2 = 1] = \frac{15}{18}, \quad P[v_i^2 = -3] = \frac{2}{18}, \]
\[ P[v_i^2 = -9] = \frac{1}{18}, \quad \forall k \geq 1. \]

To describe the uncertainty of each sensor according to our theoretical model we have considered two independent sequences of independent Bernoulli variables, \( \{\theta_i^k; k \geq 0\}, \quad i = 1, 2 \), with constant probabilities, \( P[\theta_i^k = 1] = \theta_i \), and defined the Bernoulli variables
\[ \gamma_i^k = 1 - \theta_i^k - \theta_i^{k-1} \theta_i^0, \quad i = 1, 2. \]

Note that if \( \gamma_i^0 = 0 \), then \( \theta_i^{k-1} = 1 \) and \( \theta_i^k = 0 \), and hence, \( \gamma_i^{k+1} = 1 \). This fact guarantees that the signal cannot be missing in two successive observations. So, the considered observation equation covers those signal transmission models with stand-by sensors, in which any failure in the transmission is immediately detected and the old sensor is then replaced.

Since the variables \( \theta_i^k \) and \( \theta_i^0 \) are independent, \( \gamma_i^k \) and \( \gamma_i^0 \) are also independent for \( |k-s| \geq 2 \). The mean of these variables is \( p^s = 1 - \theta_i (1 - \theta_i) \) and
\[ E[(\gamma_i^k - p^s)(\gamma_i^s - p^s)] = \begin{cases} 0, & |k-s| \geq 2 \\ -(1-p^s)^2, & |k-s| = 1 \\ p^s(1-p^s), & k = s. \end{cases} \]

To analyze the performance of the proposed estimators, the linear and quadratic filtering error variances have been calculated for different values of \( \theta_1 \) and \( \theta_2 \) which provide different values of the probabilities \( p_1^s \) and \( p_2^s \). Since \( p^s \) are the same if the value \( 1 - \theta_i \) is considered instead of \( \theta_i \); more specifically, the values \( \theta_1 = 0.1, 0.2, 0.3, 0.4, 0.5 \) (which lead to \( p^s = 0.91, 0.84, 0.78, 0.76, 0.75 \), respectively) have been used.

In all the cases examined, the filtering error variances present insignificant variation from the 5th iteration onwards and, consequently, only the error variances at a specific iteration are shown here. Figure 1 displays the linear and quadratic filtering error variances at \( k = 50 \) versus \( \theta_1 \) (for constant values of \( \theta_2 \)) and Figure 2 shows these variances versus \( \theta_2 \) (for constant values of \( \theta_1 \)). From these figures it is gathered that, as \( \theta_1 \) or \( \theta_2 \) increase (and, consequently, the false alarm probability of the corresponding sensor, \( 1 - p^s \), and the correlation (negative) between the variables \( \gamma_i^k \) and \( \gamma_i^{k-1} \) increases), the filtering error variances become greater and, hence, worse estimations are obtained. On the other hand, both figures show that the error variances corresponding to the quadratic filter are always considerably less than those of the linear filter, thus confirming the superiority of the quadratic filter over the linear one in estimation accuracy.

![Figure 1: Linear and quadratic filtering error variances at k = 50 versus θ1 with θ2 varying from 0.1 to 0.5](image)

6 Conclusion

Linear and quadratic filtering algorithms are proposed from correlated uncertain observations coming from
multiple sensors with different uncertainty characteristics. This is a realistic assumption in situations concerning sensor data that are transmitted over communication networks where, generally, multiple sensors with different properties are involved. The uncertainty in each sensor is modelled by a sequence of Bernoulli variables which are correlated at consecutive sampling times. A real application of such observation model arises for example in signal transmission problems where any failure in a sensor is immediately detected and the old sensor is replaced, thus avoiding the possibility of missing signal in two successive observations.

The algorithms are derived by applying the innovation technique to suitably-defined augmented state and observation vectors, and the LS quadratic estimator of the state is obtained from the LS linear estimator of the augmented state based on the augmented observations.

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