Application of Adomian decomposition method to nonlinear oscillators

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Abstract: In this paper, the Adomian decomposition method is applied to nonlinear oscillators. The obtained frequency and period are of high accuracy which are valid for the whole solution domain. Comparison of the obtained solution with exact solution is also given.

Key–Words: Adomian decomposition method; Nonlinear oscillator; Ball-bearing oscillator; Mathematical pendulum; Vibrations of eardrum.

1 Introduction

The study of nonlinear oscillators is of great interest to many researchers and various methods have been proposed, for example, variational iteration method [1,2], homotopy perturbation method [3,4]. In this paper we will apply the Adomian decomposition method (ADM) to determine period, frequency and approximate solution of nonlinear oscillators.

Comparison with Variational iteration method reveals that the approximate solution obtained by Adomian decomposition method converge to exact solution faster than those of Variational’s method and the period and frequency obtained by proposed method are as the same as those obtained by Variational iteration method.

2 Adomian decomposition method

The ADM developed by Adomian in 1984 [5]. It has been modified by Wazwaz in 1999. This method is based on the search for a solution in the form of a series and on decomposing the nonlinear operator into a series in which the terms are calculated recursively using the Adomian polynomials. ADM has been used by many authors to be a powerful mathematical tool for solving various types of nonlinear problems. The main properties of the method are that, it can find wide application in non-linear problems without linearization or small perturbations.

Consider the differential equation

\[ Lu + Nu = g(t) \]  

Where \( L \) is a linear operator and \( N \) is a nonlinear operator. The ADM gives the solution in a series form

\[ u(x) = \sum_{n=0}^{\infty} u_n(x) \]  

Where the components \( u_n \) are determined recurrently.

The nonlinear operator \( F(u) \) can be decomposed into an infinite series of polynomials given by

\[ F(u) = \sum_{n=0}^{\infty} A_n \]  

Where \( A_n \) are called Adomian polynomials of \( u_0, u_1, \ldots, u_n \) and defined by

\[ A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} (F[\sum_{i=0}^{n} \lambda^i u_i]), \quad n = 0, 1, 2, \ldots \]  

3 Examples

Consider the following nonlinear oscillators [6]

Example 1 (Mathematical pendulum). Many of the mathematical methods employed in non-linear oscil-
Substituting (2) and (3) into (11) gives

\[ A \cos \omega t + \frac{1}{20} A^5, \]

with initial conditions

\[ u(0) = A, \quad u'(0) = 0. \]

In order to apply the Adomian decomposition method to solve the above problem, the approximation \( u \approx u - \frac{1}{6} u^3 + \frac{1}{120} u^5 \) is used, as a result, we rewrite (5) in the following form

\[ u'' + \omega^2 u - \frac{1}{6} \omega^2 u^3 + \frac{1}{120} \omega^2 u^5 = 0, \] (6)

The differential operator \( L \) is

\[ L = \frac{\partial^2}{\partial t^2}, \] (7)

The inverse \( L^{-1} \) given by

\[ L^{-1}(\cdot) = \int_0^t \int_0^t (\cdot) dt \, dt \] (8)

Then the operator form of (6) is

\[ Lu = -\omega^2 u + \frac{1}{6} \omega^2 u^3 - \frac{1}{120} \omega^2 u^5 \] (9)

Now with applying \( L^{-1} \) on both sides (9), we have

\[ u(t) = \phi - \omega^2 L^{-1} u + \frac{1}{6} \omega^2 L^{-1} u^3 - \frac{1}{120} \omega^2 L^{-1} u^5 \] (10)

We begin with the initial approximation \( u_0(t) = A \cos \omega t \), where \( \alpha \) is an unknown constant and \( \omega \) is the angular frequency of the system. Substituting the initial approximation into (10)

\[ u(t) = A \cos \omega t - \omega^2 L^{-1} u + \frac{1}{6} \omega^2 L^{-1} u^3 - \frac{1}{120} \omega^2 L^{-1} u^5 \] (11)

Substituting (2) and (3) into (11) gives

\[ \sum_{n=0}^{\infty} u_n(t) = A \cos \omega t - \omega^2 L^{-1} \sum_{n=0}^{\infty} u_n + \frac{1}{6} \omega^2 L^{-1} \left( \sum_{n=0}^{\infty} A_n \right) \] (12)

The Adomian polynomials that represent the nonlinear term \( u^3 - \frac{1}{20} u^5 \), given by

\[ A_0 = u_0^3 - \frac{1}{20} u_0^5 \]
\[ A_1 = 3u_0^2 u_1 - \frac{1}{4} u_0^4 u_1 \]
\[ A_2 = 3u_0 u_2^2 + 3u_0^2 u_2 - \frac{1}{2} u_0^3 u_2^2 - \frac{1}{4} u_0^4 u_2 \]

By using the recurrence relation, we find

\[ u_0(t) = A \cos \omega t \]

and

\[ u_n(t) = -\omega^2 L^{-1} u_{n-1} + \frac{1}{6} \omega^2 L^{-1} A_{n-1}, \quad n \geq 1. \]

so

\[ u_1(t) = \frac{1}{27000} \alpha^2 \left[ -27000 + 3500 \alpha^2 - 149 \alpha^4 \right] \cos \omega t \]
\[ +3000 \alpha^2 \cos \omega t + 9 \alpha^4 \cos^3 \omega t \]
\[ +20 \alpha^4 \cos^3 \omega t + 120 \alpha^4 \cos \omega t \]

Then the first order approximation is

\[ u_0(t) + u_1(t) = \left[ A + \frac{A}{\alpha^2} - \frac{1}{8} \frac{A^3}{\alpha^2} + \frac{1}{192} \frac{A^5}{\alpha^2} \right] \cos \omega t \]
\[ -\frac{A}{\alpha^2} + \frac{7}{54} \frac{A^3}{\alpha^2} - \frac{149}{27000} \frac{A^5}{\alpha^2} \]
\[ +\left[ -\frac{1}{216} \frac{A^3}{\alpha^2} + \frac{1}{3456} \frac{A^5}{\alpha^2} \right] \cos 3 \omega t \]
\[ +\frac{1}{48000} \frac{A^5}{\alpha^2} \cos 5 \omega t \] (13)

In order to solve it’s first order approximation in as simple a manner as possible, we equate the coefficient of \( \cos \omega t \) equal to zero by setting

\[ \alpha = \sqrt{1 - \frac{1}{8} \frac{A^2}{\alpha^2} + \frac{1}{192} \frac{A^4}{\alpha^2}} \] (14)

We, therefore, obtain it’s first order approximation

\[ u_0(t) + u_1(t) = \left[ A + \frac{A}{\alpha^2} + \frac{7}{54} \frac{A^3}{\alpha^2} - \frac{149}{27000} \frac{A^5}{\alpha^2} \right] \cos \omega t \]
\[ +\left[ -\frac{1}{216} \frac{A^3}{\alpha^2} + \frac{1}{3456} \frac{A^5}{\alpha^2} \right] \cos 3 \omega t \]
\[ +\frac{1}{48000} \frac{A^5}{\alpha^2} \cos 5 \omega t \] (15)
with $\alpha$ defined as (11).

It’s period can be expressed as follows:

$$T_{app} = \frac{2\pi}{\alpha \omega} = \frac{2\pi}{\omega \sqrt{1 - \frac{1}{8}A^2 + \frac{1}{102}A^4}}$$  \hspace{1cm} (16)

Which as the same as that obtained by Variational iteration method [1].

The approximate period is of high accuracy, for example, when $A = \frac{\pi}{2}$, the value obtained from (16) is $T = 1.17 T_0$, while the exact one is $T_{ex} = 1.16 T_0$, where $T_0 = \frac{2\pi}{\omega}$.

**Example 2** (A ball-bearing oscillator). In this example we consider the motion of a ball-bearing oscillating in a smooth tube that is bent into a curve such that the restoring force depends upon the cube of the displacement. The governing equation, which can be readily obtained, is

$$u'' + \varepsilon u^3 = 0,$$  \hspace{1cm} (17)

with initial conditions

$$u(0) = A, \quad u'(0) = 0.$$  

The differential operator $L$ is

$$L = \frac{\partial^2}{\partial t^2}$$  \hspace{1cm} (18)

The inverse $L^{-1}$ given by

$$L^{-1}(\cdot) = \int_0^t \int_0^t (\cdot) dt \, dt$$  \hspace{1cm} (19)

Then the operator form of (17) is

$$Lu = -\varepsilon u^3$$  \hspace{1cm} (20)

Now with applying $L^{-1}$ on both sides (20), we have

$$u(t) = \phi - \varepsilon u^3$$  \hspace{1cm} (21)

We begin with the initial approximation $u_0(t) = A \cos \omega t$, where $\omega$ is the angular frequency of the system. Substituting the initial approximation into (21)

$$u(t) = A \cos \omega t - \varepsilon L^{-1}u^3$$  \hspace{1cm} (22)

Substituting (2) and (3) into (22) gives

$$\sum_{n=0}^{\infty} u_n(t) = A \cos \omega t - \varepsilon \sum_{n=0}^{\infty} A_n$$  \hspace{1cm} (23)

The Adomian polynomials that represent the nonlinear term $u^3$, given by

$$A_0 = u_0^3$$  $$A_1 = 3u_0^2 u_1$$  $$A_2 = 3u_0 u_1^2 + 3u_0^2 u_2$$  $$\ldots$$

By using the recurrence relation, we find

$$u_0(t) = A \cos \omega t$$

and

$$u_n(t) = -\varepsilon L^{-1}A_{n-1}, \quad n \geq 1.$$  

So

$$u_0(t) = A \cos \omega t$$

$$u_1(t) = \frac{\varepsilon A^3}{\omega^2} \left[ -\frac{7}{9} + \frac{1}{36} \cos 3\omega t + \frac{3}{4} \cos \omega t \right]$$

Then the first order approximation is

$$u_0(t) + u_1(t) = \left[ \frac{4A\omega^2 + 3\varepsilon A^3}{4\omega^2} \right] \cos \omega t$$

$$+ \frac{\varepsilon A^3}{9 \omega^2} \left[ -7 + \frac{1}{4} \cos 3\omega t \right]$$  \hspace{1cm} (24)

Where the angular frequency $\omega$ is identified with the physical understanding that no secular terms should be appeared in (24), which leads to

$$\omega = \frac{\sqrt{3}}{2} \varepsilon^{\frac{1}{7}} A$$  \hspace{1cm} (25)

We, therefore, obtain it’s first order approximation

$$u_0(t) + u_1(t) = \frac{\varepsilon A^3}{9 \omega^2} \left[ -7 + \frac{1}{4} \cos 3\omega t \right]$$  \hspace{1cm} (26)

with $\omega$ defined as (25). The approximate period can be expressed in the form

$$T_{app} = \frac{2\pi}{\omega} = \frac{4\sqrt{3}\pi}{3\varepsilon^{\frac{1}{7}} A} = 7.25\varepsilon^{-\frac{1}{7}} A^{-1}$$  \hspace{1cm} (27)

Which as the same as that obtained by Variational iteration method [7].

The approximate period is of high accuracy, while the exact period is $T_{ex} = 7.4163 \varepsilon^{-\frac{1}{7}} A^{-1}$.

So the results obtained by the present theory are valid for $A > 0$ and $\varepsilon > 0$. 

**Example 2** (A ball-bearing oscillator). In this exam-
Example 3 (Vibrations of eardrum). As a third example, we consider the equation of the motion of the human eardrum

\[ u'' + \omega^2 u + \varepsilon u^2 = 0, \quad (28) \]

with initial conditions

\[ u(0) = A, \quad u'(0) = 0. \]

The differential operator \( L \) is

\[ L = \frac{\partial^2}{\partial t^2} \]

The inverse \( L^{-1} \) given by

\[ L^{-1}(\cdot) = \int_{0}^{t} \int_{0}^{t} (\cdot) dt \]

Then the operator form of (17) is

\[ Lu = -\omega^2 u - \varepsilon u^2 \quad (31) \]

Now with applying \( L^{-1} \) on both sides (20), we have

\[ u(t) = \phi - \omega^2 u - \varepsilon u^2 
\]

We begin with the initial approximation \( u_0(t) = A \cos \alpha \omega t \), where \( \alpha(\varepsilon) \) is a non-zero constant with \( \alpha(0) = 1 \). Substituting the initial approximation into (32)

\[ u(t) = A \cos \alpha \omega t - \omega^2 L^{-1} u - \varepsilon L^{-1} u^2 \]

Substituting (2) and (3) into (33) gives

\[ \sum_{n=0}^{\infty} A_n(t) = A \cos \alpha \omega t - \omega^2 L^{-1} \sum_{n=0}^{\infty} A_n \]

The Adomian polynomials that represent the nonlinear term \( u^2 \), and given by

\[ A_0 = u_0^2 \]

\[ A_1 = 2u_0 u_1 \]

\[ A_2 = u_1^2 + 2u_0 u_2 \]

\[ A_3 = 2u_1 u_2 + 2u_0 u_3 \]

\[ \ldots \]

By using the recurrence relation, we find

\[ u_0(t) = A \cos \alpha \omega t \]

and

\[ u_n(t) = -\omega^2 L^{-1} u_{n-1} - \varepsilon L^{-1} A_{n-1}, \quad n \geq 1. \]

so

\[ u_0(t) = A \cos \alpha \omega t \]

\[ u_1(t) = \frac{A}{4\alpha^2 \omega^2} \left[ -4\omega^2 - \varepsilon A + 4\omega^2 \cos \alpha \omega t + \varepsilon A \cos^2 \alpha \omega t - \varepsilon A \omega^2 t^2 \right] \]

Then the first order approximation is

\[ u_0(t) + u_1(t) = \left[ A + \frac{A}{\alpha^2} \right] \cos \alpha \omega t \]

\[ -\frac{A}{\alpha^2} + \frac{\varepsilon A^2}{4\omega^2} \left[ -1 + \cos^2 \alpha \omega t - \omega^2 t^2 \right] \]

(35)

To obtain an approximation with more high accuracy, we should identify the unknown \( \alpha \) such that in the next iteration secular terms will not occur, so the coefficient of \( \cos \alpha \omega t \) in (35) must vanish, i.e.

\[ \alpha = \pm i \quad (36) \]

We, therefore, obtain it’s first order approximation

\[ u_0(t) + u_1(t) = A - \frac{\varepsilon A^2}{4\omega^2} \left[ -1 + \cos^2 \alpha \omega t + \omega^2 t^2 \right] \]

(37)

With \( \alpha \) defined as (36).

Comparison with Variational iteration method reveals that the the first order approximation obtained by proposed method converge to it’s exact solution faster than those of Variational’s method [1].

4 Conclusion

The Adomian decomposition method, which is proved to be a powerful mathematical tool to nonlinear oscillators, can be easily extended to any nonlinear oscillator, and the present short note can be used as paradigms for many other applications in searching for period, frequency and approximate solutions of various nonlinear oscillators.

References


