TOPOLOGICAL PROPERTIES FOR THE TRANSLATION OF A NON-LINEAR TOPOLOGY

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Abstract: We present some characterizations of T1, T2 separation and metrizability for the translation of an almost linear topology.

Key-words: almost linear space, almost linear topological space, translation of a topology.

1 Introduction

Let $\Gamma$ be a field of scalars and $X$ a linear space over $\Gamma$. We denote by $\mathcal{P}(X)$ the family of nonempty subsets of $X$. On $\mathcal{P}(X)$ the algebraic operations $(A, B) \mapsto A + B$ and $(\lambda, A) \mapsto \lambda A$, with $\lambda \in \Gamma$, verify the most axioms from the definition of the linear space, excepting the existence of the symmetrical element and the distributivity with respect to the sum of scalars. So on $\mathcal{P}(X)$ it is obtained a non-linear structure. This notion was called almost linear space (a.l.s.) by the author in [1] and [2], but it is also known as semi-linear space (see, for example, [11]). Godini names another similar notion by almost linear space ([10]). In the sequel, for continuity in terminology we use the term almost linear space (a.l.s.). Another studies on classical operations with subsets can be found in [9].

If $X$ is also a topological space, we must endow $\mathcal{P}(X)$ with a hyperspatial topology and we ask us that this hypertopology is compatible with the operations of a.l.s. The answer is in the affirmative for the linear topology $\tau_L$ ([5]), lower and upper Hausdorff topologies $\tau_H^+$ and $\tau_H^-$, lower Vietoris topology $\tau_V^-$ and proximal topology $\tau_P$ (see [1]) and [2]). These examples have suggested us to introduce ( in [1]) the notion of almost linear topological space (a.l.t.s.).

We notice that for an almost linear topology, a fundamental system of neighbourhoods for a point $x_0$ isn’t, generally, the translation with $x_0$ of a fundamental system of neighbourhoods for the origin. So we introduce in [2] a new notion, namely the translation of a topology on an a.l.t.s.

The aim of this paper is to give some characteristic properties of separation and metrizability for the translation of a topology on an a.l.t.s.

In the Section 2 we recall certain notions, notations and results which we need in this work; we describe some hypertopologies and we give some known theorems on the translation of a topology.

Section 3 is dedicated to the results of T1 and T2-separation and metrizability and to some examples.

2 Terminology and notations

Definition 2.1. Let $L$ be a nonvoid set and

- $+$ : $L \times L \rightarrow L$ and $-$ : $\Gamma \times L \rightarrow L$ two operations on $L$ (with $\Gamma$ a field of scalars) which satisfy the axioms:

$S1) (x + y) + z = x + (y + z), \forall x, y, z \in L;$
there exists an unique element 0 ∈ L such that 

S2) there exists an unique element 0 ∈ L such that 

S2) there exists an unique element 0 ∈ L such that 

S3) x + y = y + x, ∀x, y ∈ L; 

S4) λ(μx) = (λμ)x, ∀x, μ ∈ Γ, ∀x ∈ L; 

S5) 1 · x = x, ∀x ∈ L; 

S6) λ(x + y) = λx + λy, ∀x ∈ Γ, ∀x, y ∈ L. 

We say that (L, +, ·) is an almost linear space (denoted by a.l.s.).

Definition 2.2. Let (L, +, ·) an a.l.s.; the structure 
(L, +, ·, σ) is called almost linear topological space (or a.l.t.s.) if the operations "+", "·": L × L → L and "\": Γ × L → L are both continuous in the topology σ.

Remark 2.1 (see [2, p.8]). If \( V(0) \) is a family of subsets of L satisfying the following axioms:

(V0) 0 ∈ V for any V ∈ \( V(0) \); 

(V1) \( ∀V_1, V_2 ∈ V(0) \) \( ∃V_3 ∈ V(0) \) such that \( V_3 ≤ V_1 \cap V_2 \); 

(V2) \( ∀V ∈ V(0) \) \( ∃V_1 ∈ V(0) \) such that \( V_1 + V_1 ≤ V \),

then we can generate on L a new topology τ in which a fundamental system of neighbourhoods of a point x is given by the construction

(2.1) \( U(x) = \{ U ⊆ L; ∀V ∈ V(0) \text{ such that } x + V ⊆ U \} \).

Definition 2.3. Let σ be a topology on an a.l.t.s. L and \( \forall(0) \) a fundamental system of neighbourhoods of the origin (which verifies axioms (V0) - (V2)). Then the topology τ on L given by the relation (2.1) is called the translation of the topology σ.

For details on the a.l.t.s., a.l.s. and the translation of an almost topology see [2]. Let us present some examples of a.l.s. and a.l.t.s.

We consider X a linear normed space, w the weak topology on X and denote by \( P(X) \) the family of nonempty subsets of X. We also denote:

\( Pb(X) = \{ A ∈ P(X); A is a bounded subset \} \),

\( K(X) = \{ A ∈ P(X); A is a compact subset \} \),

\( K^w(X) = \{ A ∈ P(X); A is a w-compact subset \} \) and \( D(X) = \{ A ∈ P(X); A is an open subset of X \} \).

Example 2.1. \( P(X) \) with usually operations with subsets forms an a.l.s. Obviously we have \( P(X) ⊆ Pb(X) ⊆ K(X) ⊆ K^w(X) \), then \( Pb(X), K(X), K^w(X) \) are also a.l.s.

On X we consider \( S(a, ε) = \{ x ∈ X; ∥ a - x ∥ < ε \} \) the ball of center \( a ∈ X \) and radius \( ε > 0 \) and \( B(a, ε) = \{ x ∈ X; ∥ a - x ∥ ≤ ε \} \) the closed ball of center \( a ∈ X \) and radius \( ε > 0 \).

Also \( S_ε(A) \) is the notation for \( ε-\)enlargement of \( A : S_ε(A) = \{ x ∈ X; ∃ a ∈ A \text{ such that } ∥ a - x ∥ < ε \} \) with \( A ≤ X, ε > 0 \).

Definitions 2.4. 1) The Hausdorff topology \( τ_H \) is defined on \( P(X) \) by \( τ_H = τ_H^- \lor τ_H^+ \), where a basic neighbourhoods of a set \( A_0 ∈ P(X) \) is, respectively:

\( τ_H^- \) : \( U_-(A_0, ε) = \{ A ∈ P(X); A_0 ≤ S_ε(A) \} \), with \( ε > 0 \),

and in \( τ_H^+ \) : \( U_+(A_0, ε) = \{ A ∈ P(X); A ≤ S_ε(A_0) \} \), with \( ε > 0 \).

This topology is also induced by the extended-valued semi-metric \( H_d \) on \( P(X) \), where \( H_d(A, B) = \text{sup} \{ d(a, A) - d(x, B); x ∈ X \} \). Equivalently, \( H_d(A, B) = \text{max} \{ e(A, B), e(B, A) \} \), where \( e(a, B) = \text{sup} \{ d(a, B); a ∈ A \} \) is the Hausdorff excess of \( A \) with respect to \( B \).

3) The lower Vietoris topology \( τ^- \) on \( P(X) \) is given by the following subbase:

\( V^- = \{ A ∈ P(X); A ∩ V \neq ∅ \} \),

where \( V \) is an open subset of \( X \).

Examples 2.2. The following spaces are almost linear topological:

\( (P(X), +, ·, τ^-) \), \( (Pb(X), +, ·, τ_H^-) \), \( (Pb(X), +, ·, τ_H^+) \) and \( (Pb(X), +, ·, τ_H) \) and \( (Pb(X), +, ·, τ_P) \).

3 Separation and metrizability

In the following, our purpose is to characterize the separations T1 and T2 on the a.l.t.s.

Theorem 3.1. Let L be an a.l.s., σ a topology on L that satisfies the axioms (V0) - (V2) and its translated τ. Then, (L, τ) is a T1 separate space if and only if one of the following equivalent properties is fulfilled:

(3.1) If \( x, y ∈ L \) such that, for any \( V ∈ V(0) \) there exists \( v ∈ V \) having the property: \( x = y + v \), then \( x = y \); 

(3.2) \( \bigcap_{V ∈ V(0)} V = \{ 0 \} \), where \( V(0) \) is an arbitrary fundamental system of neighbourhoods of the origin.

Proof. One can use the fact that a topological space L is a T1 separate space if and only if the singletons are closed sets.

This is similar with the following condition:

\( ∃ x ∈ L \) and \( ∀ y ∈ L \) such as, for \( V ∈ V(0) \) one has

\( (y + V) ∩ \{ x \} = ∅ \implies y = x \),
from where the (3.1) form derives.
This is also equivalent to:
\[(3.3) \quad \text{if } x, y \in L \text{ for which } x \in y + \bigcap_{V \in \mathcal{V}(0)} V,\]
then \(x = y.\)

Obviously, (3.2) implies (3.3).
Conversely, let \(a \in \bigcap_{V \in \mathcal{V}(0)} V\) (a nonempty set: contains \(0\)).
Then, \(x = y + a \text{ and } x = y, \text{i.e., } x = x + a; \text{ from the uniqueness of element } 0\), postulated by axiom S2, Definition 2.1, it follows that \(a = 0.\)

**Theorem 3.2.** If \(\tau\) is the translated of a topology on an a.l.s. \(L\), then \((L, \tau)\) is a T2 separate space if and only if the following condition is fulfilled:
\[(3.4) \quad \text{if } x, y \in L \text{ such as } x + V_1 = y + V_2, \text{ then } x = y.\]

**Proof.** Let \(x, y \in L.\) If \(x \neq y,\) then there is \(V_1, V_2 \in \mathcal{V}(0)\) such as \((x + V) \cap (y + V) = \emptyset.\) From the axiom (V1), for \(V_1\) and \(V_2,\) there exists \(V \in \mathcal{V}(0)\) such as \(V \subseteq V_1 \cap V_2.\) Then, \((x + V) \cap (y + V) = \emptyset.\)

Rephrasing, according to the converse’s contrary, it follows that, if \(x, y \in L\) such as \((x + V) \cap (y + V) \neq \emptyset,\) for any \(V \in \mathcal{V}(0),\) then \(x = y, \text{i.e., } (3.4).\)

**Remark 3.1.** The topological condition (3.4) allows us to extract equal elements from an equality relationship, without using the symmetrical elements, thus ‘supplying’ the existence axiom, for each element of \(L,\) of its symmetric.

Thus, one can give a theorem for an a.l.s. metrizability.

**Theorem 3.3.** Let \(L\) be an a.l.s., \(\sigma\) a topology on \(L\) and \(\mathcal{V}(0) = (U_k)_{k \in \mathbb{N}^+}\) a countable, fundamental system of neighbourhoods of the origin, in the topology \(\sigma.\) If \(\mathcal{V}(0)\) satisfies the axioms (V0) – (V2) and the condition
\[(3.2)’ \quad \bigcap_{k \in \mathbb{N}^+} U_k = \{0\}\]

Then the \(L\) space with the translated topology is metrizable.

**Proof.** One can closely follow the classical proof for the metrizability of linear topological spaces (see, for example, [15]):

Based on axiom (V2), one can recurrently construct a fundamental system of neighborhoods of the origin \((V_n)_{n \in \mathbb{N}}\) for the \(\sigma\) topology of \(L,\) having the following property:
\[V_{n+1} \cap V_{n+1} \cap V_{n+1} \subseteq V_n, \text{ for any } n \in \mathbb{N}.\]

Let \(\tau\) be the topology translated on \(L.\)

We consider the function \(g : L \times L \rightarrow \mathbb{R},\)
\[g(x, y) = \inf \{\|x - v\|: x \in y + V_n\}.\]

If \(x, y, u, v \in L\) and \(\varepsilon > 0,\) such as
\[g(x, u) < \varepsilon, g(u, v) < \varepsilon, g(v, y) < \varepsilon,\]
then \(g(x, y) < 2\varepsilon.\)

We define \(d : L \times L \rightarrow \mathbb{R},\)
\[d(x, y) = \inf \sum_{i=0}^{n-1} g(u_i, u_{i+1}),\]
where infimum is considered on all the finite systems of points \((u_i)_{i \in \mathbb{T}}\) for which \(u_0 = x \text{ and } u_p = y.\)

Then, the double inequality takes place:
\[\frac{1}{2} g(x, y) \leq d(x, y) \leq g(x, y), \forall x, y \in X.\]

It results that \(d\) is a metric on \(L.\) The topology induced by the metric \(d\) is equivalent with \(\tau,\) because \(x + V_k \subseteq B_d(0, \frac{1}{2}) \subseteq x + V_{k+1}, \text{ for any } x \in L.\)

**Remark 3.2.**
(i) The metric \(d\) constructed in the proof of the Theorem 3.3 satisfies the following condition of "semi-invariance" to translations:
\[d(x + z, y + z) \leq d(x, y), \forall x, y, z \in L,\]

as the function \(g\) defined above fulfills the same inequality: if \(x \in y + V_k,\) then for any \(z \in L,\) one has \(x + z \in y + z + V_k,\) thus
\[g(x + z, y + z) \leq g(x, y).\]

(ii) The family
\[\left\{x + B_d\left(0, \frac{1}{2^k}\right)\right\}_{k \in \mathbb{N}^+}\]
also constitutes a fundamental system of neighborhoods for \(x\) on the topology \(\tau.\) One can notice that, if \(d\) is any metric on \(L,\) the sets \(x + B(0, \varepsilon)\) and \(B(x, \varepsilon)\) are not necessarily comparable. But if \(d\) is "semi-invariant" to translations, from the inequality \(d(x + u, x) \leq d(u, 0)\) applied to the elements \(u \in B(0, \varepsilon)\) one can find that
\[x + B(0, \varepsilon) \subseteq B(x, \varepsilon).\]

(iii) If we renounce at the hypothesis of \(T_1\) separation from the Theorem 3.3, the almost linear topology will be only semi-metrizable.

In the following, we will apply the Theorem 3.3 for the topologies \(\tau_H^-, \tau_H^+, \tau_V\) and \(\tau_P,\) in order to find the conditions for semimetrizability. We will design as \(D(X)\) the family of nonempty open sets of linear normed space \(X.\)

**Corollary 3.4.**
(i) The translated topology \(\tau_H^-\) is semimetrizable on \(\mathcal{P}b(X);\)

(ii) The translated of the topology \(\tau_H^+\) is semimetrizable on \(D(X);\)

(iii) The translated of the topology \(\tau_V\) is semimetrizable on \(\mathcal{P}(X);\)

(iv) The translated of the topology \(\tau_P\) is semimetrizable on \(D(X).\)
Proceedings of the 8th WSEAS Int. Conf. on NON-LINEAR ANALYSIS, NON-LINEAR SYSTEMS AND CHAOS

i) A fundamental system $\mathcal{V}$ of neighbourhoods of the origin in $\tau^+_H$ on $\mathcal{P}b(X)$ will be formed by the sets having the form

$$V^+_H(0; B, \varepsilon) = \{ A \in \mathcal{P}b(X); B \subseteq S_\varepsilon(A) \}$$

with $B \in \mathcal{P}b(X)$ containing the origin and $\varepsilon > 0$.

Let be the family $\mathcal{V}'$ of the neighbourhoods for the origin of $\tau^-_H$ having the type

$$V^+_H(0; B; 0, p, \frac{1}{n})$$

with $p, n \in \mathbb{N}^*$. The system $\mathcal{V} \subseteq \mathcal{V}'$ is countable and it defines the same topology as $\mathcal{V}$, as: $B$ is bounded, then there exists $p \in \mathbb{N}^*$ such that $B \subseteq B(0, p)$; we take $n = \lfloor 1/\varepsilon \rfloor + 1$ (where $[\alpha]$ is the greatest integer smaller than the real number $\alpha$). Then for any $A \in \mathcal{P}b(X)$ having the property: $B(0, p) \subseteq S_{1/n}(A)$, the following inclusion is also valid:

$$B \subseteq S_\varepsilon(A) \quad \left( \text{so } V^+_H(0; B; 0, p, \frac{1}{n}) \subseteq V^+_H(0; B, \varepsilon) \right).$$

ii) For the topology $\tau^+_H$ on $\mathcal{D}(X)$, a fundamental system $\mathcal{V}$ of neighbourhoods of the origin is formed by sets of the type:

$$V^+_H(0; B, \varepsilon) = \{ A \in \mathcal{D}(X); A \subseteq S_\varepsilon(B) \},$$

where $B \in \mathcal{D}(X)$ with $0 \in B$ and $\varepsilon > 0$.

For such a set $B$, there exists $p \in \mathbb{N}^*$ such as $S(0, 1/p) \subseteq B$, and if $n = \lfloor 1/\varepsilon \rfloor + 1$, then for any $A \in \mathcal{D}(X)$ such that $A \subseteq S_{1/n}(S(0, 1/p))$, it follows that $A \subseteq S_\varepsilon(B)$, since $n, p \in \mathbb{N}^*$ also constitutes a fundamental system of neighbourhoods equivalent to $\mathcal{V}$.

iii) Let be $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$, with $n \in \mathbb{N}^*$ and $U^-_{\mathcal{V}} = S(0, \varepsilon_1)^- \cap S(0, \varepsilon_2)^- \cap \ldots \cap S(0, \varepsilon_n)^- = S(0, \varepsilon)^-$,

where $\varepsilon = \min \{ \varepsilon_j; j = \lfloor 1/n \rfloor \}$, is a fundamental neighbourhood of the origin in $\tau^-_H$.

By choosing for every $\varepsilon > 0$ a $n \in \mathbb{N}^*$ i.e., $n = \lfloor 1/\varepsilon \rfloor$, one can obtain

$$S(0, 1/n)^- \subseteq S(0, \varepsilon)^-,$$

that is, the family

$$\mathcal{V} = \{ S(0, 1/n)^-; n \in \mathbb{N}^* \}$$

is contained in $\mathcal{V} = \{ S(0, \varepsilon)^-; \varepsilon > 0 \}$; also $\mathcal{V}$ represents a fundamental system of neighbourhoods for the origin in $\tau^-_H$.

iv) This follows from ii) and iii), as $\tau_P = \tau^-_V \vee \tau^+_H$.

Finally, we offer a sufficient condition for a metric $d$, in order to induce a topology which is almost linear.

Theorem 3.5. Let $(L, +, \cdot)$ be an a.l.s. and $d$ a semi-metric on $L$, satisfying the properties:

1) $d(a + c, b + c) \leq d(a, b)$, for any $a, b, c \in L$,

2) $d(\lambda a, \lambda b) = |\lambda| d(a, b)$, for any $a, b, \lambda \in L$, $\lambda \in \Gamma$

3) $d(\lambda a, \mu a) \leq |\lambda - \mu| \cdot d(a, 0)$, for any $\lambda, \mu \in \Gamma$ and any $a \in L$.

Then, the topology induced on $L$ by the semi-metric $d$ is almost linear.

Proof. Let $x_0, y_0 \in L$ and $(x_n)_{n \in \mathbb{N}^*}, (y_n)_{n \in \mathbb{N}^*} \subseteq L$, with $d(x_n, x_0) \to 0, d(y_n, y_0) \to 0$.

One has the inequalities:

$$d(x_n + y_n, x_0 + y_0) \leq d(x_n + y_n, x_0 + y_n) + d(x_0 + y_0, x_0 + y_0) \leq d(x_n, x_0) + d(y_n, y_0),$$

hence $d(x_n, x_0) \to 0$ and $\lambda_n \to \lambda$ in $\Gamma$.

In this case,

$$d(\lambda_n x_n, \lambda_0 x_0) \leq d(\lambda_n x_n, \lambda_n x_0) + d(\lambda_n x_0, \lambda_0 x_0) \leq |\lambda_n - \lambda_0| \cdot d(x_n, 0) + |\lambda_n - \lambda_0| \cdot d(x_0, 0),$$

Consequently, $d(\lambda_n x_n, \lambda_0 x_0) \to 0$.

Remark 3.3. As an example, the above conditions I, II, III are fulfilled by the semi-metric $H$ on the family $\mathcal{P}b(X)$ (see [1], Proposition 4.5).

References:


