A Permeability Function for Brinkman’s Equation

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Abstract: Inlet flow conditions to a two-dimensional channel filled with a porous material are developed when the flow through the channel is governed by Brinkman’s equation. The presence of solid, impermeable channel walls gives rise to significant viscous shear effects that can be handled through the introduction of a smooth, variable permeability function. For the sake of comparison, inlet profiles are derived for the cases of flow with constant permeability and flow with variable permeability.

Key-Words: -Brinkman’s Equation, Permeability Function, Inlet Profile

1 Introduction
A number of models have been developed over the past fifteen decades to describe flow through porous media (cf. [2], [3], [5], [8] and the references therein). The celebrated Darcy’s law expresses the mean average velocity (superficial velocity) as proportional to the driving pressure gradient. In its simplest form, Darcy’s law can be expressed as:

\[- \nabla p - \frac{\mu}{k} \vec{v} = 0 \]  ...(1)

where \( k \) is the permeability, \( p \) is the pressure, \( \mu \) is the viscosity of the fluid, and \( \vec{v} \) is the velocity vector field.

Eq. (1) is marked by the absence of both inertial and viscous shear terms, and accounts for fluid viscosity effects through a viscous damping term. The model is valid for low-permeability flows, where changes at the micro-scale are insignificant compared to a large, macro-scale [3], [8]. Furthermore, the model does not account for any viscosity increase due to the solid matrix, neither does it account for the presence of a macroscopic boundary [4], [10]. The absence of a macroscopic cross-stream flow in Darcy’s regiment results in an apparent slip condition even at an impermeable boundary. Its low differential order does not properly handle viscous shear effects that inevitably arise in the presence of solid, macroscopic boundary bounding sparsely-packed, high porosity porous material of finite depth.

Darcy’s law, however, provides a connection between the mean filter velocity, the driving pressure gradient and the characteristics of the medium (namely, the permeability). Furthermore, it provides a practical definition of permeability that is important in the study of macromolecular systems, [10].

A comprehensive model that describes flow through porous media, and incorporates most of the existing classifications of flow models, takes the following form:

\[ \rho \left( \beta (\eta - 1) + 1 \right) (\vec{v} \cdot \nabla) \vec{v} = - \nabla p + \mu (\beta (\eta - 1) + 1) \nabla^2 \vec{v} \]

\[ - \beta \left( \frac{\mu}{k} \vec{v} + \frac{\sigma}{k} |\vec{v}| \vec{v} \right) \]  ...(2)

where \( \vec{v} \) is the velocity vector, \( p \) is the pressure, \( k \) is the constant permeability, \( \mu \) is the fluid viscosity, \( \mu_e \) is the effective viscosity of the fluid in the porous medium, \( \eta = \frac{\mu_{eff}}{\mu} \), \( \beta \) is a binary parameter whose value is 0 if
the flow is in free-space and 1 if flow is in a porous medium, $\sigma$ is the form drag coefficient associated with the Forchheimer equation, $\nabla$ is the gradient operator, and $\nabla^2$ is the Laplacian operator.

For different choices of the parameters in equation (2), we recover forms of momentum equations that govern fluid flow in different regiments. Of particular interest to the current work are the following forms:

a) When $\beta = 0$ and $\vartheta = 1$, the Navier-Stokes equations are obtained.

b) When $\beta = 1$, $\eta = 0$, $\vartheta = 0$, and $\sigma = 0$, equations (2) reduce to Darcy’s law.

c) When $\beta = 1$, $\eta = 1$, $\vartheta = \mu_{eff} / \mu$, and $\sigma = 0$, equations (2) reduce to Darcy-Lapwood-Brinkman equation.

Note that if we take $\eta = 0$ in case (c), we recover the well-known Brinkman’s equation, [2], namely

$$-\nabla p + \mu_{eff} \nabla^2 \vec{v} - \frac{\mu}{k} \vec{v} = 0$$

(3)

Brinkman, [2], introduced a correction to Darcy’s law to account for viscous shear effects and the changes in viscosity that arise due to the introduction of the solid, porous matrix. Clearly, equation (3) reduces to Darcy’s law (equation (1)) when $\mu_{eff} = 0$.

While Brinkman, [2], may have preferred the choice of $\mu = \mu_{eff}$, a number of authors have stressed that the two viscosity coefficients could be different depending on the porous material, [4].

Through volume averaging, Whitaker, [9], concluded that the viscous shear term in Brinkman’s equation is only significant in the presence of a solid, macroscopic boundary, on which a no-slip condition is imposed. A number of authors emphasized the importance of a viscous shear term to account for the boundary layer that develops when a viscous fluid flows past a porous layer or near a solid wall, [6], [7], [8]. Nield, [5], argued that if the Brinkman term is to be included, then porosity variations should be accounted for. Sahraoui and Kaviany, [6], suggested the use of a variable permeability model when the viscous shear term is important. It is this latter concept of permeability variation that is of interest in the current work, as summarized in what follows.

There are many industrial applications that involve the presence of a solid wall bounding a finite porous medium. These include flow though pipe filled with porous material; flow through porous layers terminated by solid surfaces; and lubrication configurations lined by porous lining, to name only a few. For these and various other applications it is imperative that we understand the mechanism in which the macroscopic boundary is taken into account.

In the study of flow through porous pipes and channels, we need to properly impose the entry conditions to the configurations. This mandates a need for velocity profiles that are compatible with Brinkman’s equation, and a continuously defined permeability function that better describes the flow through porous layers.

If one considers the Navier-Stokes flow through a two-dimensional channel, a Poiseuille profile is used to describe the parabolic, parallel flow entry condition to the channel. This profile is compatible with the two-dimensional Navier-Stokes equations. In the current work, we derive an entry condition that satisfies the Brinkman equation and takes into account permeability variation that is necessary for the validity of Brinkman’s equation. Furthermore, and similar to the Navier-Stokes case where the entry profile is tied to the velocity profile, we seek to tie permeability variation and channel entry condition to the velocity profile through the porous medium.

2 Analysis

There are two points of interest in using Brinkman’s equation. The first is the viscosity variation due to the introduction of the porous solid matrix. This is accounted for through the use of two viscosity coefficients in the Brinkman equation. The second point is associated with the sudden drop to zero of the permeability on the macroscopic solid boundary. In a way, this is associated with porosity discontinuity as we move from the medium to the macroscopic boundary. Ideally, the porous medium should have a different compaction near the macroscopic boundary so that porosity reduction is a continuous function.

This situation may be explained in terms of a (rectangular) sponge that is saturated with a viscous fluid. If the sponge is bounded by solid plates and squeezed, the material is compacted near the solid walls and throughout the sponge in a gradual manner that impedes or reduces the flow near the plates and allows a
maximum flow along the median line axially through the sponge. What is being generated here is a permeability-stratification where the permeability is continuously stratified, and the fluid moves from the low permeability to the high permeability layers. Avoiding discontinuity in the permeability mandates the introduction of smooth variation (reduction) in the permeability as we move closer and closer to the solid boundary. Tied to the defined smooth permeability is a velocity profile that must be compatible with Brinkman’s equation.

In order to illustrate this concept, we consider Poiseuille flow through a porous channel of a large-enough length, bounded on either side by solid, impermeable walls, as shown in Fig. 1.

![Fig. 1. Dimensional Representative Sketch](image_url)

On the solid walls, a no-slip velocity condition is imposed. We assume that the channel is “squeezed” to the point that permeability is continuously varying and the channel depth is \( D \). The idea here is to cast the governing equation in a dimensionless form in order to arrive at a permeability definition at the inlet to the channel that overcomes permeability discontinuity.

The flow is described by Brinkman’s equation, and is pressure-driven. For parallel flow, the normal velocity component vanishes and equation (3) reduces to:

\[
\frac{d^2 u}{dy^2} + \frac{\mu}{\mu_{eff} k(y)} u = -C
\]  

where

\[
C = -\frac{1}{\mu_{eff}} \frac{dp}{dx}
\]

If the permeability \( k(y) \) is constant (say \( k \)), then an inlet velocity profile that satisfies the no-slip condition at the solid walls, and is compatible with Brinkman’s equation, is obtained by directly solving equation (4). The profile takes the form, [1]:

\[
u(y) = a_1 \exp \left( -\frac{\mu}{k_{eff} y} \right) + a_2 \exp \left( \frac{\mu}{k_{eff} y} \right) + \frac{C k_{eff}}{\mu}
\]

where

\[
a_1 = -a_2 - \frac{C k_{eff}}{\mu}
\]

and

\[
a_2 = \frac{-1 + \exp \left( -D \sqrt{\frac{\mu}{k_{eff}}} \right) C k_{eff}}{\exp \left( D \sqrt{\frac{\mu}{k_{eff}}} \right) - \exp \left( -D \sqrt{\frac{\mu}{k_{eff}}} \right)}
\]

If the permeability is assumed to be a function of \( y \), then we try to answer the following questions:

1) What is an appropriate form of \( k(y) \)? and
2) What is the corresponding inlet profile?

Assuming that the permeability is a quadratic, parabolic function of \( y \), with a value of zero at the upper and lower walls, then a maximum value of permeability is reached along the horizontal centerline of the channel \( (y = D/2) \). It should be noted that this choice is for illustration purposes only; other continuous forms of the permeability function are possible. Now, letting

\[
k(y) = ay^2 + by + c
\]

and using the conditions \( k(0) = k(D) = 0 \), we obtain \( c = 0, \ b = -aD; \) hence,

\[
k(y) = a(y^2 - Dy).
\]

Since \( k(y) \) reaches a maximum, say \( k_{max} \), at \( y = D/2 \), we see that

\[
k_{max} = -\frac{aD^2}{4}.
\]
The general form of \( k(y) \) is thus given by
\[
k(y) = -\frac{4k_{\text{max}}}{D^2} (y^2 - Dy)
\] ...(12)
and equation (4) becomes:
\[
\frac{d^2 u}{dy^2} + \frac{\mu D^2}{4\mu_{\text{eff}} k_{\text{max}} (y^2 - Dy)} u = -C.
\] ...(13)

Now, introducing the dimensionless variables:
\[
Y = \frac{y}{D}; K(Y) = \frac{k(y)}{k_{\text{max}}},
\]
we obtain the following dimensionless permeability function:
\[
K(Y) = 4(Y - Y^2).
\] ...(14)

The channel becomes dimensionless of unit depth, and equation (13) takes the following form:
\[
\frac{d^2 u}{dY^2} + \frac{\mu D^2}{4\mu_{\text{eff}} k_{\text{max}} (Y^2 - Y)} u = -D^2 C.
\] ...(15)

Assuming \( u \) to be a parabolic function of \( Y \), given by:
\[
u(Y) = A(Y^2 - Y)
\] ...(16)
and using equation (16) in equation (15), we obtain:
\[
A = -\frac{4D^2 C k_{\text{max}}}{8k_{\text{max}} + D^2 \mu / \mu_{\text{eff}}}.
\] ...(17)

Hence
\[
u(Y) = -\frac{4D^2 C k_{\text{max}}}{8k_{\text{max}} + D^2 \mu / \mu_{\text{eff}}}(Y^2 - Y).
\] ...(18)

Since \( u \) attains its maximum value at \( Y = \frac{1}{2} \), then
\[
u_{\text{max}} = \frac{D^2 C k_{\text{max}}}{8k_{\text{max}} + D^2 \mu / \mu_{\text{eff}}}
\]
\[
= \frac{k_{\text{max}} D^2 C}{k_{\text{max}} [8 + D^2 \mu / (k_{\text{max}} \mu_{\text{eff}})]} = \frac{D^2 C}{8 + H}
\] ...(19)

where
\[
H = \frac{D^2 \mu}{k_{\text{max}} \mu_{\text{eff}}} = \frac{D^2}{9k_{\text{max}}}
\] ...(20)
is a dimensionless number.

Defining \( U = \frac{u}{\nu_{\text{max}}} \), we obtain the dimensionless form of \( U \) as:
\[
U(Y) = 4(Y - Y^2).
\] ...(21)

Note that at the inlet to the channel, \( K(Y) \) has a maximum value of 1, but \( k(y) \) has a maximum value of \( k_{\text{max}} \). Furthermore, the inlet velocity \( U(Y) \) has a maximum value of 1, while the dimensional inlet velocity \( u(y) \) has a maximum value \( u_{\text{max}} \) which depends on the channel height, \( D \), the viscosity coefficients \( \mu \) and \( \mu_{\text{eff}} \), the pressure gradient, and the maximum permeability \( k_{\text{max}} \). In other words, \( u_{\text{max}} \) is a function of the driving pressure gradient and the dimensionless number \( H \).

3 Comments on the \( H \) Number
The \( H \) number appearing in equation (20) is a dimensionless number whose value depends on the channel depth, \( D \), the maximum permeability in a given porous medium, and the ratio \( \vartheta = \frac{\mu_{\text{eff}}}{\mu} \). Knowledge of \( H \) facilitates knowledge of \( u_{\text{max}} \), as given in equation (19), for a given pressure gradient. Clearly, determination of \( H \) is tied to the knowledge of maximum permeability for a given porous medium and a
knowledge of the ratio \( q = \frac{\mu_{\text{eff}}}{\mu} \). If \( \mu_{\text{eff}} = \mu \), then \( H \) is simply \( \frac{D^2}{k_{\text{max}}} \).

Neale and Nader [4] reported some values for the ratio \( \frac{\mu_{\text{eff}}}{\mu} \). For the sake of illustration, if we take their reported values on permeability to be \( k_{\text{max}} \) in our analysis, then we can generate the following Table (Table 1) for \( H \) in terms of the porous medium thickness.

<table>
<thead>
<tr>
<th>Material</th>
<th>( k_{\text{max}}; \text{cm}^2 )</th>
<th>( \frac{\mu_{\text{eff}}}{\mu} )</th>
<th>( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foametal</td>
<td>( 8.2 \times 10^{-4} )</td>
<td>( 16 )</td>
<td>( \frac{10^2 D^2}{1.312} )</td>
</tr>
<tr>
<td>Aloxite</td>
<td>( 1.6 \times 10^{-5} )</td>
<td>( 0.01 )</td>
<td>( \frac{10^7 D^2}{1.6} )</td>
</tr>
</tbody>
</table>

Table 1. Values of \( H \) based on reported permeability and \( \frac{\mu_{\text{eff}}}{\mu} \) (cf. [4]).

4 Application to Flow over Porous Layers

We will use the variable permeability model developed in this work to study the flow through a channel terminated by a finite porous layer. In particular, we derive an expression for the velocity at the interface between the channel and the porous layer and show its dependence on the dimensionless number \( H \), and on the flow and fluid parameters.

Consider the flow of a viscous fluid through a straight channel of depth \( D \) bounded by a porous layer of thickness \( D \), as shown in Fig. 2.

The channel is bounded by a solid wall at \( y = D \), and the porous layer is terminated from below by a macroscopic boundary (solid wall) at \( y = -D \). Flow through the configuration is driven by a constant pressure gradient, \( -p_x \), and is governed by the Navier-Stokes equations, which take the form

\[
\frac{\partial u}{\partial y} = \frac{1}{\mu} p_x \quad \ldots(22)
\]

where \( u = u(y) \) is the velocity in the channel.

Flow through the porous layer is governed by the following form of Brinkman’s equation that is valid for variable permeability porous layer:

\[
\frac{\partial v}{\partial y} = \frac{1}{\mu_{\text{eff}} k(y)} \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial y} \right) = \frac{1}{\mu_{\text{eff}}} p_x \quad \ldots(23)
\]

where \( v \) is the velocity in the porous layer.

Along the sharp interface, \( y = 0 \), between the channel and the porous layer, the flow satisfies the following matching conditions (velocity and shear stress continuity):

\[
u(0) = v(0) = u_i \quad \ldots(24)
\]

where \( u_i \) is the unknown velocity at the interface.

\[
u_y(0) = \frac{\mu_{\text{eff}}}{\mu} v_y(0) \quad \ldots(25)
\]

Equations (22) and (23) are to be solved subject to (24) and (25), and subject to no-slip on solid boundaries, namely:

\[
u(D) = 0 \quad \ldots(26)
\]

\[
u(-D) = 0 \quad \ldots(27)
\]

Permeability in the porous layer is assumed to be a variable function \( k(y) \), such that it falls to zero on the lower wall and reaches a maximum at the sharp interface, namely

\[
k(-D) = 0 \quad \ldots(28)
\]

\[
k(0) = k_{\text{max}} \quad \ldots(29)
\]

Assume that \( k(y) \) is a parabolic function of the form

\[
k(y) = ay^2 + b \quad \ldots(30)
\]

Then, using (28) and (29), equation (30) takes the form

\[
k(y) = k_{\text{max}} \left( 1 - \frac{y^2}{D^2} \right) \quad \ldots(31)
\]

Using (31) in (23), we obtain the following equation governing flow through the porous layer:
\[ v_{yy} - H \frac{v}{(D^2 - y^2)} = \frac{1}{\mu_{\text{eff}}} p_x \]  
\[ \text{...(32)} \]

where \( H \) is given by (20).

Now, equations (22) and (32) are to be solved subject to conditions (24) and (25).

Solution to (22) takes the form:
\[ u = \frac{1}{\mu} p_x \frac{y^2}{2} + c_1 y + c_2 \]  
\[ \text{...(33)} \]

where \( c_1 \) and \( c_2 \) are constants to be determined.

Using (24) and (26) we obtain \( c_2 = u_i \), and

\[ c_1 = -\frac{D p_x}{2 \mu} \frac{u_i}{D} \], and (33) takes the form
\[ u = \frac{1}{\mu} p_x \frac{y^2}{2} - \left( \frac{D p_x}{2 \mu} + \frac{u_i}{D} \right) y + u_i. \]  
\[ \text{...(34)} \]

An expression for the shear stress is obtained from (34) as:
\[ u_y = \frac{1}{\mu} p_x y - \left( \frac{D p_x}{2 \mu} + \frac{u_i}{D} \right). \]  
\[ \text{...(35)} \]

At \( y=0 \), equation (35) reduces to:
\[ u_y(0) = -\frac{D p_x}{2 \mu} - \frac{u_i}{D}. \]  
\[ \text{...(36)} \]

Now, in the absence of a general solution to equation (32), assume the velocity profile in the porous layer to be of the form:
\[ v = \alpha y^2 + \beta y + \gamma \]  
\[ \text{...(37)} \]

where \( \alpha, \beta, \gamma \) are constants to be determined.

An expression for the shear stress is obtained from (37) as:
\[ v_y = 2 \alpha y + \beta. \]  
\[ \text{...(38)} \]

At \( y=0 \), equation (38) reduces to:
\[ v_y(0) = \beta. \]  
\[ \text{...(39)} \]

Now, using condition equations (36) and (39) in condition (25), we obtain:
\[ \beta = -\frac{D p_x}{2 \mu_{\text{eff}}} - \frac{\mu}{\mu_{\text{eff}}} \frac{u_i}{D}. \]  
\[ \text{...(40)} \]

Using (24) in (37), we obtain \( \gamma = u_i \). With the knowledge of \( \beta \) and \( \gamma \), using condition (27) in (37) yields
\[ \alpha = -\frac{p_x}{2 \mu_{\text{eff}}} \left( \frac{\mu}{\mu_{\text{eff}} D^2} + \frac{1}{D^2} \right) u_i. \]  
\[ \text{...(41)} \]

Equation (37) then takes the form
\[ v = \left[ -\frac{p_x}{2 \mu_{\text{eff}}} - \left( \frac{\mu}{\mu_{\text{eff}} D^2} + \frac{1}{D^2} \right) u_i \right] y^2 \]  
\[ - \left[ \frac{D p_x}{2 \mu_{\text{eff}}} + \frac{\mu}{\mu_{\text{eff}} D} \right] y + u_i \]  
\[ \text{...(42)} \]

Now the velocity profile (42) must satisfy (32) at the interface, \( y = 0 \). At \( y = 0 \), equation (32) reduces to:
\[ v_{yy}(0) - H \frac{v(0)}{D^2} = \frac{1}{\mu_{\text{eff}}} p_x. \]  
\[ \text{...(43)} \]

With the knowledge that \( v(0) = u_i \) and, from (42), we have
\[ v_{yy}(0) = 2 \alpha = -\frac{p_x}{\mu_{\text{eff}}} \left( \frac{2 \mu}{\mu_{\text{eff}} D^2} + \frac{2}{D^2} \right) u_i \]  
\[ \text{...(44)} \]

then equation (43) yields the following expression for the velocity at the interface between the channel and the porous layer:
\[ u_i = -\frac{D^2 p_x}{\mu + \mu_{\text{eff}} (1 + H/2)}. \]  
\[ \text{...(45)} \]

Equation (45) provides a positive value for \( u_i \) since \( p_x < 0 \) and all other quantities in (45) are positive. Furthermore, it shows the dependence of \( u_i \) on the dimensionless number \( H \) (equivalently, on \( k_{\text{max}} \) as given by equation (20)), the driving pressure gradient.
the coefficients of viscosity and the channel depth and the depth of the porous layer.

5 Two-Dimensional Flow Equations

In this section, we will discuss how the dimensionless number $H$ enters in the vorticity-streamfunction formulation of flow equations for variable permeability porous media. For flow governed by Brinkman’s equation, $H$ may serve as a replacement for flow Reynolds number.

Consider the flow of a viscous fluid through a two-dimensional porous domain where Brinkman’s equation (3) is valid. For incompressible fluid, the flow is also governed by the equation of continuity which takes the form:

$$u_x + v_y = 0. \quad \ldots (46)$$

Continuity equation implies the existence of a streamfunction, $\psi$, defined in terms of the velocity components $u$ and $v$ by

$$u = \psi_y \quad \ldots (47)$$

$$v = -\psi_x. \quad \ldots (48)$$

Vorticity, $\omega$, is expressed in terms of the velocity components as:

$$\omega = \nabla \times \vec{v} = v_x - u_y. \quad \ldots (49)$$

Using (47) and (48) in (49), we obtain the streamfunction equation

$$\psi_{xx} + \psi_{yy} = -\omega. \quad \ldots (50)$$

Taking the curl of equation (3), we obtain the following vorticity equation for flow in a porous medium with constant permeability

$$\omega_{xx} + \omega_{yy} - \frac{\mu}{k\mu_{eff}} \omega = 0. \quad \ldots (51)$$

Equations (50) and (51) are the vorticity-streamfunction form of the governing equations when permeability is constant.

If permeability is taken as a function of $y$, then taking the curl of equation (3) results in the following vorticity equation

$$\omega_{xx} + \omega_{yy} - \frac{\mu}{k\mu_{eff}} \omega + \frac{\mu}{\mu_{eff}} u \frac{d}{dy} \left( \frac{1}{k} \right) = 0. \quad \ldots (52)$$

Equations (50) and (51) can be cast in dimensionless variables, with respect to a characteristic length, $D$, and a characteristic velocity, $U_\infty$, to yield a stream-function equation of the same form as (50), but with a vorticity equation of the form

$$\Omega_{xx} + \Omega_{yy} - \frac{\text{Re}^*}{\text{Re} K} \Omega = 0 \quad \ldots (53)$$

The dimensionless permeability is defined by

$$K = \frac{k}{D^2} \quad \ldots (54)$$

and the Reynolds numbers are defined as follows:

$$\text{Re} = \frac{\rho U_{\infty} D}{\mu} \quad \ldots (55)$$

and

$$\text{Re}^* = \frac{\rho U_{\infty} D}{\mu_{eff}} \quad \ldots (56)$$

where $\rho$ is the density of the fluid.

Rather than dealing with the question of how high a Reynolds number can be when dealing with a variable permeability model, we define the following dimensionless variables for the vorticity equation:

$$X = \frac{x}{D}; Y = \frac{y}{D}; U = \frac{u}{k_{max}}; V = \frac{v}{u_{max}} \quad \text{and} \quad \Omega = \frac{\omega D}{u_{max}}. \quad \ldots (57)$$

Applying (57) to (52) results in the following dimensionless vorticity equation

$$\Omega_{xx} + \Omega_{yy} - \frac{H}{K} \Omega + HU \frac{d}{dy} \left( \frac{1}{K} \right) = 0 \quad \ldots (58)$$

where $H$ is given by equation (20).

Clearly, $H$ depends on the values of the viscosities, a characteristic length $D$ (typically the depth of a channel) and on the maximum permeability for a given porous medium. $H$ is inversely proportional to $k_{max}$, for given values of $\mu$, $\mu_{eff}$ and $D$. In the limit as $k_{max} \to \infty$, $H \to 0$. The use of $H$ to characterize flow through variable permeability media may prove to be more efficient than the use of Reynolds number.

While this analysis is valid for Brinkman’s equation, the general model governing flow through variable permeability medium may be analyzed in the same manner. In fact, it can be shown that analysis of equation (2), which involves inertial terms, will lead to an equation involving $H$ as a dimensionless number. In addition, using the dimensionless variables defined in (57) leads to another dimensionless of the form
\[ \frac{\rho Du_{\text{max}}}{\mu_{\text{eff}}} \]  \hspace{1cm} \ldots(59)

This dimensionless number is the same as \( \text{Re}^* \) defined in (56), with the characteristic velocity being \( u_{\text{max}} \). If \( u_{\text{max}} \) is expressed by an equation similar to (19), one may have a knowledge of how high \( \text{Re}^* \) can be taken.

6 Conclusions

In the case of pressure driven, Poiseuille flow through porous media, an inlet velocity profile has been properly introduced so that it reflects the dependence of the velocity on medium and fluid properties. The profile is compatible with Brinkman’s equation, and is associated with a choice of a smooth permeability function that overcomes permeability discontinuity as we move closer to the solid, bounding walls. It is recommended that the developed profile and permeability function (or variations thereof) be adopted in the analysis of Brinkman-type flow through pipes and channels.

A dimensionless number, \( H \), has been introduced in the analysis. This number may prove to be useful in the analysis of variable permeability media. It may also be of some use in estimating maximum velocity in a given porous medium. Some estimated values of \( H \) are given in Table 1 based on suggested values of the ratio of effective viscosity to fluid velocity. We have used the introduced concepts to find an expression of the velocity at the interface between a channel and a bounding porous layer. In addition, we considered the vorticity-streamfunction formulation of variable permeability media in an attempt to shed some light on the role the dimensionless number \( H \) plays. We have shown how \( H \) may be of value in avoiding an answer to the question of how high the speed of the flow can be taken when Brinkman’s equation is the governing equation of flow through porous media with variable permeability.

References: