Abstract: Field equations governing the steady flow of an incompressible micro-polar fluid through isotropic porous sediments are derived using intrinsic volume averaging. The model equations might be of applicability to the study of lubrication problems in configurations involving porous linings, and to the study of polymer and oil flow through porous structures.

Key-Words: Micropolar Fluid, Porous Media, Intrinsic Volume Averaging

1 Introduction

Theory of micro-polar fluids was introduced by Eringen [5] in an attempt to model the flow of non-Newtonian fluids exhibiting microscopic effects that arise from the micro-rotational motion and spin inertia which enable them to support couple stress and body couples. Over the last four decades, the theory has been applied to the study of various physical and biological applications, including lubrication, the flow of blood in animal tissues, heat transfer, and boundary layer analysis, (cf. [1], [6], [9], [10], [13], [14]). More recently, there has been a renewed interest in application of micro-polar fluid flow through micro-channels [11]. This has been motivated in part by the need to design micro-fluidic devices involving micro-channels with small dimensions [11].

For these and many other applications, together with a more complete literature survey, one is referred to the work of Lukaszewicz [12].

The motion of a micro-polar fluid in free-space is described by a rigid-motion velocity vector and a micro-rotation velocity vector that accounts for the spin of fluid elements, in a given volume element, about the centroid of the volume element. In the absence of energy and heat transfer effects, the usual governing equations are composed of the conservation of mass equation, and a set of coupled linear and angular momentum equations. Solution to the governing equations in enclosures or in the presence of solid boundaries requires imposing appropriate boundary conditions.

Typical of these is the usual no-slip flow condition of a viscous fluid and the assumption of no-spin there on geometrically-described solid boundary. This situation becomes formidable when the micro-polar fluid flows through a porous structure, due to the complexity of the pore geometry and the absence of a mathematical description of the solid matrix.

In order to circumvent this, we attempt to develop a set of field equations governing the flow of a micro-polar fluid through an isotropic porous structure using the method of intrinsic volume averaging. This procedure has gained popularity since the introduction of averaging theorems, (cf. [2], [3], [4] and the references therein), and has been successfully implemented in deriving various models of flow through porous media.

The models developed in this work might find applications in the study of high density and high viscosity polymers and oils in porous configurations, and in the study of lubrication problems in configurations possessing porous linings.

2 Governing Equations

The flow of a micro-polar fluid in free space is governed by the equation of continuity, the linear momentum equation, and the angular momentum equation [5]. When the fluid is incompressible, and body forces and body couple are absent, the equations governing steady-state flow can be written in the following form:
\( \nabla \cdot \vec{V} = 0 \) \hspace{1cm} \ldots(1)

\[
\rho (\vec{V} \cdot \nabla) \vec{V} = -\nabla p + (\mu + \tau) \nabla^2 \vec{V} + 2\tau (\nabla \times \vec{G}) \]
\hspace{1cm} \ldots(2)

\[
\rho k^2 (\vec{V} \cdot \nabla) \vec{G} = 2\tau \nabla \times \vec{V} + (\alpha + 2\beta) \nabla (\nabla \cdot \vec{G}) - (\beta + \gamma) \nabla \times (\nabla \times \vec{G}) - 4\tau \vec{G} \]
\hspace{1cm} \ldots(3)

where \( \vec{V} \) is the velocity vector field, \( \vec{G} \) is the micro-rotation velocity vector, \( p \) is the pressure, \( \rho \) is the fluid density, \( r \) is the radius of gyration of a fluid element, in a given volume element, about the centroid of the volume element, and \( \alpha, \beta, \gamma, \mu, \) and \( \tau \) are viscosity coefficients.

Equation (2) can be written in the following dyadic form that is suitable for volume averaging:
\[
\rho \nabla \times (\nabla \times \vec{G}) = \nabla (\nabla \cdot \vec{G}) - \nabla^2 \vec{G} \]
\hspace{1cm} \ldots(4)

Making use of the vector identity
\[
\nabla \times (\nabla \times \vec{G}) = \nabla (\nabla \cdot \vec{G}) - \nabla^2 \vec{G} \]
\hspace{1cm} \ldots(5)

and using equation (1), we write equation (3) in the dyadic form:
\[
\rho k^2 \nabla \times \vec{V} \vec{G} = 2\tau \nabla \times \vec{V} + (\alpha + \beta - \gamma) \nabla (\nabla \cdot \vec{G}) + (\beta + \gamma) \nabla^2 \vec{G} - 4\tau \vec{G} \]
\hspace{1cm} \ldots(6)

In order to develop a set of field equations governing the flow of a micro-polar fluid through an isotropic porous medium, the equations governing the flow in free space, \( i.e. \) equations (1), (4) and (6) will be averaged over a Representative Elementary Volume (REV), introduced in [2]. The effects of the porous microstructure on the flowing fluid will be accounted for through the concept of a Representative Unit Cell (RUC), introduced in [3] and [4].

Typical conditions on the velocity and spin vectors are the no-slip and no-spin assumptions on the solid matrix. These are implemented in this work and translate to: \( \vec{V} = \vec{G} = 0 \) on the stationary solid matrix.

### 3 The Averaging Approach
Following Bachmat and Bear, [2], a Representative Elementary Volume, REV, is a control volume that contains fluid and porous matrix in the same proportion as the whole porous medium. In other words, it is a control volume whose porosity is the same as that of the whole porous medium. The porosity, \( \varphi \), is defined as the ratio of the pore volume to the bulk volume of the medium. In terms of the REV, porosity is defined as \( \varphi = \frac{V_p}{V} \), where \( V_p \) is the pore volume within the REV, which contains the fluid, and \( V \) is the bulk volume of the REV. In terms of microscopic and macroscopic length scales, \( l \) and \( L \) respectively, the REV is chosen such that \( l^3 \ll V \ll L^3 \).

In order to develop the equations of flow through a porous structure we define the volumetric phase average of a quantity \( F \), as:
\[
< F > = \frac{1}{V} \int \varphi F dV . \]
\hspace{1cm} \ldots(7)

The intrinsic phase average (that is, the volumetric average of \( F \) over the effective pore space, \( V_\varphi \)) is defined as:
\[
< F > _\varphi = \frac{1}{V_\varphi} \int \varphi F dV . \]
\hspace{1cm} \ldots(8)

Relationship between the volumetric phase average and the intrinsic phase average can be seen from equations (7), and (8), and the definition of porosity, as:
\[
< F > = \varphi < F > _\varphi . \]
\hspace{1cm} \ldots(9)

The following averaging theorems are then applied to equations (1), (4), and (6). Letting \( F \) and \( H \) be volumetrically additive scalar quantities, \( \vec{\psi} \) a vector quantity, and \( c \) a constant (whose average is itself), then, [3], [4]:
\[
(i) \ldots < c F > = c < F > = c < F > _\varphi .
\]
\[
(ii) \ldots < \nabla F > = \varphi \nabla < F > _\varphi + \frac{1}{V} \int S \nabla \cdot \vec{n} dS
\]

where \( S \) is the surface area of the solid matrix in the REV that is in contact with the fluid, and \( \vec{n} \) is the unit normal vector pointing into the solid. The quantity \( F^\varphi = F - < F > \) is the deviation of the averaged quantity from its true (microscopic) value.
Due to the no-slip condition, a surface integral is zero if it contains the fluid velocity vector explicitly.

Due to the no-spin condition, a surface integral is zero if it contains the spin vector explicitly.

4 Averaging the Governing Equations

4.1 The Continuity Equation

Taking the average of both sides of equation (1) and applying Rule (iii), the continuity equation takes the following intrinsic volume averaged form:

\[ \nabla \cdot \phi < \vec{V} > = \nabla \cdot \phi < \vec{V} > + \frac{1}{V} \int_{S} \nabla \cdot \vec{n} dS. \]  

For the incompressible flow at hand, continuity of mass flow translates into vanishing normal component of velocity. This can be seen by invoking the divergence theorem, and making use of the continuity equation (1), as follows:

\[ \int_{S} \vec{V} \cdot \vec{n} dS = \int_{V_{p}} \nabla \cdot \vec{V} dV = 0. \]  

Using Rule (i) and (ii), the surface integral vanishes. Accordingly, equation takes the final form:

\[ \nabla \cdot \phi < \vec{V} > = 0. \]  

4.2 Linear Momentum Equation

Taking the average of both sides of equation (4) and applying Rule (iii), we get:

\[ < \rho \nabla \cdot \vec{V} \vec{V} > = -\nabla p +< (\mu + \tau) \nabla ^{2} \vec{V} > + 2 \tau (\nabla \times \vec{G}) > \]  

Equation (13) is evaluated term by term, as follows. Using rules (i) and (v), we obtain:

\[ < \rho \nabla \cdot \vec{V} \vec{V} > = \rho \nabla \cdot \phi < \vec{V} > + \frac{\rho}{V} \int_{S} \vec{V} \vec{V} \cdot \vec{n} dS. \]  

In order to average the term \((\mu + \tau)\nabla ^{2} \vec{V} \), it is first written in the form \((\mu + \tau)\nabla \cdot \nabla \vec{V} \). Averaging Rules (i) and (iii), followed by Rule (ii), are then applied to obtain:

\[ < (\mu + \tau) \nabla ^{2} \vec{V} > = \frac{\mu + \tau}{V} \int_{S} \nabla \cdot \vec{V} \vec{V} \cdot \vec{n} dS. \]  

Using Rule (i) and (ii), the last term on the right-hand-side of equation (13) is expressed in the following form

\[ < 2 \tau (\nabla \times \vec{G}) > = 2 \tau \nabla \cdot \phi < \vec{G} > - \frac{2 \tau}{V} \int_{S} \vec{G} \times \vec{n} dS. \]
Now, using equations (14), (15), (17) and (18) in equation (13), we obtain the following intrinsic volume averaged form of the linear momentum equation:

\[
\rho \nabla \cdot \mathbf{\varphi} < \mathbf{V} >_\varphi = -\nabla \varphi < p >_\varphi + \\
(\mu + \tau) \nabla^2 \varphi < \mathbf{V} >_\varphi + 2\tau \nabla \times \varphi < \mathbf{G} >_\varphi \\
- \rho \nabla \cdot \varphi < \mathbf{V}^\ast \mathbf{V}^\ast >_\varphi - \frac{1}{V} \int_{S} \varphi < \mathbf{n} > dS + \frac{\mu + \tau}{V} \int_{S} \mathbf{V} \cdot \mathbf{n} dS \\
- \frac{\rho}{V} \int_{S} \mathbf{V} \cdot \mathbf{n} dS - \frac{2\tau}{V} \int_{S} \mathbf{G} \times \mathbf{n} dS.
\]

Using Rules (vii) and (viii), the terms \(\int_{S} \mathbf{V} \cdot \mathbf{n} dS\) and \(\int_{S} \mathbf{G} \times \mathbf{n} dS\) vanish, and (19) becomes:

\[
\rho \nabla \cdot \mathbf{\varphi} < \mathbf{V} >_\varphi = -\nabla \varphi < p >_\varphi + \\
(\mu + \tau) \nabla^2 \varphi < \mathbf{V} >_\varphi + 2\tau \nabla \times \varphi < \mathbf{G} >_\varphi \\
- \rho \nabla \cdot \varphi < \mathbf{V}^\ast \mathbf{V}^\ast >_\varphi - \frac{1}{V} \int_{S} \varphi < \mathbf{n} > dS + \frac{\mu + \tau}{V} \int_{S} \mathbf{V} \cdot \mathbf{n} dS.
\]

4.3 Angular Momentum Equation

In order to average the angular momentum equation (6), we apply rule (iii) followed by rule (i) to get:

\[
\rho k^2 < \mathbf{V} \times \mathbf{G} > = 2\tau < \mathbf{V} \times \mathbf{V} > + (\alpha + \beta - \gamma) < \nabla (\mathbf{V} \cdot \mathbf{G}) > + (\beta + \gamma) < \nabla^2 \mathbf{G} > - 4\tau < \mathbf{G} >
\]

Applying rule (v), we get:

\[
\rho k^2 < \mathbf{V} \times \mathbf{G} > = \rho k^2 \mathbf{V} \cdot \varphi < \mathbf{V} >_\varphi < \mathbf{G} >_\varphi + \rho k^2 < \mathbf{V} \mathbf{G} \cdot \mathbf{n} dS.
\]

Using Rule (vi), we obtain:

\[
< 2\tau (\mathbf{V} \times \mathbf{V}) > = 2\tau \nabla \times \varphi < \mathbf{V} >_\varphi = \frac{2\tau}{V} \int_{S} \mathbf{V} \times \mathbf{n} dS.
\]

Using Rule (ii), we obtain:

\[
(\alpha + \beta - \gamma) < \nabla (\mathbf{V} \cdot \mathbf{G}) > = \\
(\alpha + \beta - \gamma) \varphi < \nabla \mathbf{N} < \mathbf{V} \cdot \mathbf{G} >_\varphi + \\
\frac{(\alpha + \beta - \gamma)}{V} \int_{S} (\nabla \mathbf{G})^\ast \mathbf{n} dS.
\]

Following a similar procedure to that used in obtaining (16), we have:

\[
(\beta + \gamma) < \nabla^2 \mathbf{G} > = (\beta + \gamma) < \nabla^2 \varphi < \mathbf{G} >_\varphi + \\
\frac{\beta + \gamma}{V} \int_{S} \mathbf{G} \cdot \mathbf{n} dS + \frac{\beta + \gamma}{V} \int_{S} \mathbf{G} \cdot \mathbf{n} dS.
\]

Using (9), we have:

\[
4\tau < \mathbf{G} > = 4\tau \varphi < \mathbf{G} >_\varphi.
\]

Using (22)-(26) in (21), we obtain

\[
\rho k^2 \nabla \cdot \varphi < \mathbf{V} >_\varphi < \mathbf{G} >_\varphi = \\
2\tau < \nabla \times \mathbf{V} > + (\alpha + \beta - \gamma) \varphi < \nabla \varphi < \mathbf{G} >_\varphi + \\
(\beta + \gamma) < \nabla^2 \mathbf{G} > - 4\tau \varphi < \mathbf{G} >_\varphi + \\
+ \frac{\beta + \gamma}{V} \int_{S} \mathbf{G} \cdot \mathbf{n} dS - \frac{\rho k^2}{V} \int_{S} \mathbf{G} \cdot \mathbf{n} dS \\
+ \frac{\rho k^2}{V} \int_{S} \mathbf{G} \cdot \mathbf{n} dS - \frac{2\tau}{V} \int_{S} \mathbf{V} \times \mathbf{n} dS.
\]

Using Rules (vii) (viii), the terms \(\int_{S} \mathbf{G} \cdot \mathbf{n} dS\), \(\int_{S} \mathbf{V} \mathbf{G} \cdot \mathbf{n} dS\) and \(\int_{S} \mathbf{V} \times \mathbf{n} dS\) vanish, and (27) takes the form:

\[
\rho k^2 \nabla \cdot \varphi < \mathbf{V} >_\varphi < \mathbf{G} >_\varphi = \\
2\tau \nabla \times \varphi < \mathbf{V} >_\varphi + (\alpha + \beta - \gamma) \varphi < \nabla \varphi < \mathbf{G} >_\varphi + \\
(\beta + \gamma) < \nabla^2 \mathbf{G} > - 4\tau \varphi < \mathbf{G} >_\varphi + \\
- \frac{\rho k^2}{V} \int_{S} (\nabla \mathbf{G})^\ast \mathbf{n} dS + \frac{(\alpha + \beta)}{V} \int_{S} (\nabla \mathbf{G})^\ast \mathbf{n} dS.
\]
5 Analyses of the Deviation Terms and Surface Integrals

5.1 The Deviation Terms
Equations (20) and (28) represent the intrinsic volume averaged linear momentum and angular momentum equations, respectively. The deviation terms and the surface integrals in these equations contain the necessary information on the interactions between the flowing fluid and the porous structure.

The terms $\nabla \cdot \varphi < \tilde{V} \cdot \nabla >_{\varphi}$ appearing in (20), and $\nabla \cdot \varphi < \tilde{V} \cdot \tilde{G} >_{\varphi}$ appearing in (28), represent the hydrodynamic dispersion of the average velocity and average micro-rotation, respectively. Hydrodynamic dispersion through porous media is the sum of mechanical dispersion and molecular diffusion. Mechanical dispersion is due to tortuosity of the flow path within the porous microstructure, and molecular diffusion arises due to diffusion of the fluid vorticity, [8]. Now, both of the above deviation terms are inertial terms representative of mechanical dispersion. In the absence of high velocity and high porosity gradients, these terms are negligible compared with molecular diffusion. In media with high porosity gradients, it has been suggested that they may be modeled using dynamic diffusivity [8].

The linear and angular momentum equations thus take the following forms, respectively

$$\rho \nabla \cdot \varphi < \tilde{V} >_{\varphi} - \varphi \nabla < p >_{\varphi} + (\mu + \tau) \nabla^2 \varphi < \tilde{V} >_{\varphi} + 2\tau \nabla \times \varphi < \tilde{G} >_{\varphi}$$

$$- \frac{1}{V} \int_{S} \left[ (p^* - n) - (\mu + \tau) \nabla \cdot \tilde{V} - (n_p^* - \mu + \tau) \nabla \cdot \tilde{n} \right] dS = F_1 < \tilde{V} >_{\varphi} \quad \ldots(29)$$

$$\rho k^2 \nabla \cdot \varphi < \tilde{V} >_{\varphi} - \varphi \nabla < \tilde{G} >_{\varphi} = 2\tau \nabla \times \varphi < \tilde{V} >_{\varphi} + (\alpha + \beta - \gamma) \varphi \nabla \cdot \nabla \cdot \tilde{G} >_{\varphi}$$

$$+ (\beta + \gamma) \nabla^2 \varphi < \tilde{G} >_{\varphi} - 4\tau \varphi < \tilde{G} >_{\varphi} + \frac{(\alpha + 2\beta)}{V} \int_{S} (\nabla \cdot \tilde{G}) \cdot \tilde{n} dS. \quad \ldots(30)$$

5.2 The Surface Integrals

5.2.1 Surface Integral Appearing in the Linear Momentum Equation
The effects of the porous matrix on the flowing fluid occur through the portion of the surface area of the solid that is in contact with the micro-polar fluid at hand. The surface integral in (29) involves the pressure deviation term and the velocity gradient, and contains the necessary information to quantify the pressure and friction forces exerted by the porous matrix on the fluid.

Accurate evaluation of the surface integral depends on the knowledge of the porous microstructure and its accurate geometric description.

The surface integral appearing in (29) can be expressed as, [7]:

$$- \frac{1}{V} \int_{S} \left[ (p^* - n) - (\mu + \tau) \nabla \cdot \tilde{V} - (n_p^* - \mu + \tau) \nabla \cdot \tilde{n} \right] dS = F_1 < \tilde{V} >_{\varphi} \quad \ldots(31)$$

where $F_1$ is a function of $\mu, \tau, S, \varphi$, the medium tortuosity, $T$, and a friction factor, $f$. An expression for $F_1$ requires a mathematical description of the porous matrix. Du Plessis and Masliyah, [3], [4], have carried out extensive analysis on evaluating this type of surface integral which arises in modeling the flow of a viscous fluid at low Reynolds number through granular and consolidated isotropic porous media. In this work, we modify their expression [4] and express the above surface integral for a granular isotropic porous medium as:

$$- \frac{1}{V} \int_{S} \left[ (p^* - n) - (\mu + \tau) \nabla \cdot \tilde{n} - n_p^* \right] dS =$$

$$\left[ 3 f_1 (1 - T)(3T - 1) l_1^2 / T^3 \right] (\mu + \tau) < \tilde{V} >_{\varphi} \quad \ldots(32)$$

wherein $f_1$ is proportional to the product of Reynolds number and the friction factor associated with the flow of a micro-polar fluid through a pipe, and $l_1$ is a microscopic length, as given by Du Plessis and Masliyah [4] who provided the following expression for the tortuosity, $T$, of the medium, based on their concept of a Representative Unit Cell (RUC), which they defined as the minimal REV in which the average properties of the porous medium are embedded:

$$T = [1 - (1 - \varphi)^{2/3}]/\varphi. \quad \ldots(33)$$
In light of (32), the intrinsic volume averaged linear momentum equation (29) takes the following final form:

\[
\begin{align*}
\rho V \cdot \nabla < \tilde{V} >_\phi &= - \phi \nabla < p >_\phi \\
+ (\mu + \tau) \nabla^2 \phi < \tilde{V} >_\phi + 2\tau \phi \nabla \times \phi < \tilde{G} >_\phi \\
- [3f_l(1-T)(3T-1)][\mu + \tau] < \tilde{V} >_\phi 
\end{align*}
\]...(34)

5.2.2 Surface Integral Appearing in the Angular Momentum Equation

The surface integral term appearing in (30), namely

\[
\int_S \nabla \cdot (\nabla \times \tilde{G} ) dS
\]

arises in averaging the term containing \( \nabla (\tilde{V} \cdot \tilde{G}) \) in the angular momentum equation (6) when using Rule (ii), in which \( F = \nabla \cdot \tilde{G} \). This surface integral involves the deviation of \( \nabla \cdot \tilde{G} \) from its true value. Since \( \nabla \cdot \tilde{G} \neq 0 \), the deviation \( (\nabla \cdot \tilde{G})^* \) is not necessarily zero for all porous structures, and its quantification depends in part on an accurate description of the porous microstructure. Let us first re-write Rule (ii) in the phase-average form:

\[
\int_S \nabla \cdot F dS = \int_S (\nabla \cdot \tilde{G}) dS
\]

which gives the superficial average of \( F \). Now, using (35), we obtain:

\[
< \nabla (\nabla \cdot \tilde{G}) > = \nabla < \nabla \cdot \tilde{G} > + \frac{1}{V} \int (\nabla \cdot \tilde{G})^* dS. \quad \text{(36)}
\]

5.3 The Case of Divergence-Free Micro-rotation

When a micro-polar fluid flows through free-space, that is, in the absence of a porous matrix, it is customary to impose a no-slip condition on the fluid-phase velocity. Continuity of mass flow translates into vanishing normal component of velocity. This can be seen by invoking the Divergence Theorem, and making use of the continuity equation to obtain equation (11).

For the micro-rotation vector, \( \tilde{G} \), a situation arises in which a divergence-free micro-rotation is used as a simplifying assumption, namely

\[
\nabla \cdot \tilde{G} = 0. \quad \text{(39)}
\]

Applying averaging Rule (v) to equation (39) yields:

\[
\nabla \cdot \phi < \tilde{G} >_\phi + \frac{1}{V} \int \tilde{G} \cdot dS = 0. \quad \text{(40)}
\]

The surface integral \( \int_S \tilde{G} \cdot dS \) implies a vanishing normal component of micro-rotation, as can be seen by invoking the Divergence Theorem to obtain:

\[
\int_S \tilde{G} \cdot dS = \int_{\tilde{G}} \nabla \cdot \tilde{G} dV = 0. \quad \text{(41)}
\]

The divergence-free micro-rotation assumption thus leads to the averaged equation:

\[
\nabla \cdot \phi < \tilde{G} >_\phi = 0. \quad \text{(42)}
\]

However, the assumption of a divergence-free micro-rotation renders the governing equations over-determined and consisting of eight scalar equations.
(divergence-free velocity field; divergence-free micro-rotation field; three scalar linear momentum equations and three scalar angular momentum equations) in the seven unknowns $\tilde{G}$, $V$ and $p$. Furthermore, with the assumption of a divergence-free micro-rotation field, the angular momentum equation (6) for flow in free-space reduces to:

$$\rho k^2 \nabla \cdot \tilde{V}G = 2\tau \nabla \times \tilde{V} + (\beta + \gamma) \nabla^2 \tilde{G} - 4\tau \tilde{G} \quad \text{(43)}$$

which has the intrinsic volume averaged form:

$$\rho k^2 \nabla \cdot \phi \tilde{V} < \tilde{V} >_{\phi} < \tilde{G} >_{\phi} = 2\tau \nabla \times \phi \tilde{V} < \tilde{V} >_{\phi}
\quad + (\beta + \gamma) \nabla^2 \phi < \tilde{G} >_{\phi} - 4\tau \phi < \tilde{G} >_{\phi}
\quad + \left(\frac{\beta + \gamma}{V} \right) (\nabla \tilde{G}) \cdot \mathbf{n} \int \text{d}S. \quad \text{(44)}$$

In order to render the governing equations determinate, consisting of eight scalar equations in eight unknowns, it is necessary to introduce a surface pressure, $p^*$, into the angular momentum equation. It is interpreted as the surface pressure needed to maintain a vanishing normal component of the micro-rotation field. Accordingly, equation (44) is modified to take the form

$$\rho k^2 \nabla \cdot \phi \tilde{V} < \tilde{V} >_{\phi} < \tilde{G} >_{\phi} = -\phi \nabla \cdot \left( p^* \right) + 2\tau \nabla \times \phi \tilde{V} < \tilde{V} >_{\phi}
\quad + (\beta + \gamma) \nabla^2 \phi < \tilde{G} >_{\phi} - 4\tau \phi < \tilde{G} >_{\phi}
\quad - \frac{1}{V} \int (\left( p^* \right) \cdot \mathbf{n}) - (\beta + \gamma) (\nabla \tilde{G}) \cdot \mathbf{n} \mid \text{d}S. \quad \text{(45)}$$

The term $-\frac{1}{V} \int (\left( p^* \right) \cdot \mathbf{n}) \mid \text{d}S$ appearing in equation (45) is the net surface pressure force which, when combined with the surface integral containing the normal component of the micro-rotation gradient field represent the frictional force exerted by the porous matrix and affecting the micro-rotation field. This surface integral can be expressed in the form:

$$-\frac{1}{V} \int [(\left( p^* \right) \cdot \mathbf{n}) - (\beta + \gamma) (\nabla \tilde{G}) \cdot \mathbf{n}] \mid \text{d}S = F_2 < \tilde{G} >_{\phi} \quad \text{(46)}$$

where $F_2$ is a function of $\beta, \gamma, S, \phi$ the medium tortuosity, $T$, and a friction factor, $f$, associated with the flow of a micro-polar fluid in a straight pipe. Following similar analysis to that followed in obtaining equation (34), an expression for this surface integral when the flow is taken in isotropic granular media is of the form:

$$[3f_l(1-T)(3T-1)l_2^2 / T^3](\beta + \gamma) < \tilde{G} >_{\phi} \quad \text{(47)}$$

and the angular momentum equation (45) takes the following final form

$$\rho k^2 \nabla \cdot \phi \tilde{V} < \tilde{V} >_{\phi} < \tilde{G} >_{\phi} =
\quad -\phi \nabla \cdot \left( p^* \right) + 2\tau \nabla \times \phi \tilde{V} < \tilde{V} >_{\phi}
\quad + (\beta + \gamma) \nabla^2 \phi < \tilde{G} >_{\phi} - 4\tau \phi < \tilde{G} >_{\phi}
\quad - [3f_l(1-T)(3T-1)l_2^2 / T^3](\beta + \gamma) < \tilde{G} >_{\phi} \quad \text{(48)}$$

### 6 The Final Forms of the Field Equations

In light of the analyses of Section 5, above, the intrinsic volume averaged set of equations governing the flow of a micro-polar fluid through an isotropic, granular porous structure, corresponding to equations (1), (4), and (6), involves solving seven scalar equations in seven unknowns, as follows:

**Conservation of Mass:**

$$\nabla \cdot \phi \tilde{V} < \tilde{V} >_{\phi} = 0. \quad \text{(49)}$$

**Linear Momentum Equation:**

$$\rho \nabla \cdot \phi \tilde{V} < \tilde{V} >_{\phi} < \tilde{G} >_{\phi} = -\phi \nabla \cdot \left( \rho v \right) + 2\tau \nabla \times \phi \tilde{V} < \tilde{V} >_{\phi}
\quad + (\mu + \tau) \nabla^2 \phi < \tilde{G} >_{\phi} + 2\tau \nabla \times \phi < \tilde{G} >_{\phi}
\quad - [3f_l(1-T)(3T-1)l_2^2 / T^3](\mu + \tau) < \tilde{V} >_{\phi}. \quad \text{(50)}$$

**Angular Momentum Equation:**

$$\rho k^2 \nabla \cdot \phi \tilde{V} < \tilde{V} >_{\phi} < \tilde{G} >_{\phi} = 2\tau \nabla \times \phi \tilde{V} < \tilde{V} >_{\phi}
\quad + (\alpha + \beta - \gamma) \nabla \cdot \phi \tilde{G} < \tilde{G} >_{\phi}
\quad + (\beta + \gamma) \nabla^2 \phi < \tilde{G} >_{\phi} - 4\tau \phi < \tilde{G} >_{\phi}. \quad \text{(51)}$$

Furthermore, if the simplifying assumption of a divergence-free micro-rotation field is used, the averaged equations take the following form which involves eight scalar equations in eight unknowns:
Continuity Equation:
\[ \nabla \cdot \vec{V} < \vec{V} >_\varphi = 0. \] \hspace{1cm} \text{ ...(52)}

Conservation of Micro-rotation:
\[ \nabla \cdot \vec{G} >_\varphi = 0. \] \hspace{1cm} \text{ ...(53)}

Linear Momentum Equation:
\[ \rho \nabla \cdot \vec{V} < \vec{V} >_\varphi = - \varphi \nabla < p >_\varphi 
+ (\mu + \tau) \nabla^2 \varphi < \vec{V} >_\varphi + 2 \tau \nabla \times \varphi < \vec{G} >_\varphi 
- [3f_1(1-T)(3T-1)l_1^2 / T^3 ](\mu + \tau) < \vec{V} >_\varphi \] \hspace{1cm} \text{ ...(54)}

Angular Momentum Equation:
\[ \rho k^2 \nabla \cdot \vec{V} < \vec{V} >_\varphi = - \varphi \nabla < p >_\varphi 
+ 2 \tau \nabla \times \varphi < \vec{V} >_\varphi + (\beta + \gamma) \nabla^2 \varphi < \vec{G} >_\varphi 
- 4 \tau \varphi < \vec{G} >_\varphi 
- [3f_1(1-T)(3T-1)l_1^2 / T^3 ](\beta + \gamma) < \vec{G} >_\varphi \] \hspace{1cm} \text{ ...(55)}

7 Conclusion
A set of equations governing the flow of a micro-polar fluid through an isotropic, granular porous medium has been derived. Analyses have been provided for both a divergence-free micro-rotation field and one in which the divergence is non-zero. Effects of the porous matrix on the flowing fluid have been accounted for based on a description of the granular microstructure given [4]. Expressions used for the arising surface integrals are similar to those given in [4], and valid for low Reynolds number. The current model is therefore appropriate for low speed flow of a micro-polar fluid through an isotropic granular porous structure.

References