Fuzzy Technologies of Weakly Structurable Systems’ Modeling and Simulation

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Abstract: The new approach to the study of weakly structurable continuous dynamic systems (WSCDS) is presented. Different from other approaches where the source of fuzzy uncertainty in dynamic systems is expert, this approach considers time as long as an expert to be the source of fuzzy uncertainty. This notably widens the area of studied problems. All these is connected to the incomplete, imprecise, anomal and extremal processes in nature and society, where connections between the system’s objects are of subjective (expert) nature, which is caused by lack of objective information about the evolution of studied system. One of our purposes is to create scenarios describing possible evolution of WSCDS using methods developed in this paper in the framework of expert-possibilistic theory. This includes construction of algorithms of logical-possibilistic simulations of anomal and extremal process analysis. The problems of an optimization and identification of a WSCDS is presented.

Key–Words: weakly structurable dynamic system, logical-possibilistic modeling and simulation, fuzzy process describing expert reflections, identification and optimal control.

1 Introduction

In this paper the new approach to the study of weakly structurable continuous fuzzy dynamic systems is presented (weakly structurable controllable dynamic system) [1]–[4], [7]–[21]. Different from other approaches where the source of fuzzy uncertainty ([12]–[21]) in dynamic systems is expert, this approach considers time as long as an expert to be the source of fuzzy uncertainty. This notably widens the area of studied problems. All these is connected to the incomplete, imprecise, anomal and extremal processes in nature and society, where connections between the system’s objects are of subjective (expert) nature, which is caused by lack of objective information about the evolution of studied system, for example in 1) economy of developing countries, business, conflictology, sociology, medical diagnostics etc; 2) management of evacuation processes in catastrophe areas, estimation of disease spreading in epidemiological regions; 3) research of complex systems of applied physics, etc. One of our purposes is to create scenarios describing possible evolution of WSCDS using methods developed in this paper in the framework of expert-possibilistic theory. This includes construction of algorithms of logical-possibilistic simulations of anomal and extremal process adaptive control and identification problems [15]–[21].

By the participants of the paper new mathematical apparatus ([12]–[14], [19]) was created in 2002–2008, where main attention is paid to rapidly developing theory of fuzzy measures (some class of capacities) and integrals. Using the theory of fuzzy measures and integrals for construction of decision support systems is not a new idea. But we have chosen one part of this theory – extremal fuzzy measures [12]–[14], which is not much well researched. In the framework of this theory a new apparatus of extended fuzzy measures was constructed on the basis of Sugeno’s upper and lower integrals ([5], [12]–[14], [19]). Using this apparatus new fuzzy extremal models of weakly structurable dynamic system control were created ([15]–[18], [20], [21]), where fuzziness is represented in time. Here the structure of time is represented by monotone extremal classes of measurable sets ([12]–[14]). On such structures uncertainty is described by extremal fuzzy measures. The problems of fuzzy-statistics of extremal fuzzy processes: identification, filtration, optimal control are investigated. Results of research are published in articles [15]–[18], [20], [21].

Here only note that when expert describes the dynamics of complex objects and “measure” system states, where the source of uncertainty is fuzzy mea-
measurements with respect to time, it is necessary to carry out “extremal” “dual” measurements, namely, measurements in extended current and compressed future fuzzy time intervals [14].

A person who makes a decision always gives an “incomplete” prognosis about a time moment for extremal, crisis, anomalous and other situations that may occur in the future. The person (expert) who makes a decision connects all such situations with future fuzzy time moments and intervals. Clearly, his/her prognosis is of fuzzy nature and the corresponding decisions should be obtained by possibilistic-statistical analysis or, speaking more exactly, by analysis of fuzzy time intervals, for which we have constructed a new mathematical instrument ([12]–[14], [19]).

When we make decisions on the basis of our past knowledge, we recall certain facts, reference data and the like. When doing so, we perform certain “expert measurements” (“expert reflections”) of our knowledge. These measurements are connected with past time moments, which as a rule are fuzzy. Hence the results of such “measurements” may frequently be also fuzzy and these results of recollections are in the end reflected in experimental data (samples). It is understood that the source of such samples is the population of fuzzy characteristics of our knowledge. This can be explained mainly by the following two reasons: first, in terms of dynamics, moments of recollections of facts and moments of “expert measurements” are fuzzy moments; second, on frequent occasions the results of “measurements” are fuzzy. Let us illustrate this viewpoint by examples. Suppose that prior to diagnosing the disease the examining physician (expert) asked the patient to present data on his temperature in extended current and compressed future time moments and intervals. Clearly, his/her prognosis about a time moment for extended future fuzzy time intervals is the subject which will be used in next sections.

1) In the time flow process information (data) obtained by expert measurements is incomplete. The polar characteristics of such information are imprecision and uncertainty. The imprecision degree of the obtained information defines extremal fuzzy time moments, while the uncertainty degree defines algebraic structures represented in form (1) [12].

2 Description of a General Model of an Extremal Fuzzy Continuous Dynamic System (EFCDS)

Following the system approach of modeling complex systems [8] we propose the following: the time structure of fuzzy dynamic systems is represented by some space of extended extremal fuzzy measures [12]

\[ (\widetilde{IF}^T_0, \widetilde{IF}^T_1, \widetilde{IF}^T_2, \widetilde{IF}^T_3, \widetilde{IF}^T_4) \quad T = R_0^+ \]

and structure (1), where \( \widetilde{IF}^T_0 \) and \( \widetilde{IF}^T_1 \) are some extremal fuzzy measures on \( \widetilde{IF}^T_0 \equiv \widetilde{IF}^T_1 \), respectively (see [12]).

Let us start describing objects of a fuzzy dynamic system. Let \( X (X \neq \emptyset) \) be the set of states of some system to be investigated. Let \( (X, B, g) \) be the space of a fuzzy measure on the measurable space \( (X, B) \), where \( B \) is a \( \sigma \)-algebra in \( X \). Let \( U (U \neq \emptyset) \) be the set of all admissible controls (of external factors) acting on the system. Assume that controls are subjected to restrictions of uncertain character in the form of some space of a fuzzy measure \( (U, B_U, g_U) \), where \( B_U \) is the measurable space of controls, while the fuzzy measure \( g_U \) describes the restrictions imposed on controls.
Let $Y (Y \neq \emptyset)$ be the set of output states of the system under consideration, and $(Y, B_Y, g_Y)$ be the space of a fuzzy measure, which describes a fuzzy distribution of output values of the system. Note that as usual $Y$ is some transformation of the set of states of $X$.

Now let us consider the Cartesian product $X \times T$ and the space of extended composition extremal fuzzy measures ([13])

$$\left( X \times T, B \otimes B_{T^*}, B \otimes B_{T^*}, g \otimes g_{T^*}, g \otimes g_{T^*} \right),$$

which is induced by the spaces $(X, B, B, g, g)$ and $(T, B_{T^*}, B_{T^*}, g_{T^*}, g_{T^*})$.

**Definition 1**

a) A lower measurable binary fuzzy relation $\tilde{Q}_\tau \in B \otimes \overline{B}_{T^*}$ is called a future fuzzy process on the measurable states of the system (i.e., $\mu_{\tilde{Q}_\tau}(x, t)$ is a lower measurable function).

b) An upper measurable binary fuzzy relation $\tilde{Q}^* \in B \otimes \overline{B}_{T^*}$ is called a current fuzzy process on the measurable states of the system (i.e., $\mu_{\tilde{Q}^*}(x, t)$ is an upper measurable function).

c) A pair $(\tilde{Q}_\tau, \tilde{Q}^*)$ of lower and upper measurable binary fuzzy relations is called an extremal fuzzy process on the measurable states of the system (i.e., $Q^* \in B \otimes B_{T^*}$ and $Q_\tau \in B \otimes \overline{B}_{T^*}$).

d) An extremal fuzzy process (EFP) is said to be ergodic if there exist the limits $\forall x \in X$, $\lim_{t \to \infty} \mu_{\tilde{Q}_\tau}(x, t) \equiv \mu_{\tilde{A}_\tau}(x)$, $\lim_{t \to \infty} \mu_{\tilde{Q}^*}(x, t) \equiv \mu_{\tilde{A}^*}(x)$, and the limit fuzzy sets $\tilde{A}_\tau$ and $\tilde{A}^*$ are measurable $\tilde{A}_\tau, \tilde{A}^* \in \overline{B}$.

Note that (see [13]) $\forall \tau \in T, \forall x \in X$

$E_{\tilde{Q}_\tau}(x, \cdot) \in \overline{B}_{T^*}$ is a future fuzzy time interval,

$E_{\tilde{Q}^*}(x, \cdot) \in \overline{B}_{T^*}$ is a current fuzzy time interval,

$E_{\tilde{Q}_\tau}(\cdot, \tau) \in \overline{B}$ is a fuzzy state of the system, which is “measurable” in the future fuzzy time interval $[\tau, +\infty)$,

$E_{\tilde{Q}^*}(\cdot, \tau) \in \overline{B}$ is a fuzzy state of the system, which is “measurable” in the current fuzzy time interval $[0, \tau]$.

It is obvious that model “measurements” of the states of the system at a real time moment $\tau > 0$ are understood as defining pairs of measurable fuzzy sets $E_{\tilde{Q}_\tau}(\cdot, \tau), E_{\tilde{Q}^*}(\cdot, \tau) \in \overline{B}$.

For all $x \in X$, $E_{\tilde{Q}_\tau}(x, \cdot)$ and $E_{\tilde{Q}^*}(x, \cdot)$ are a current fuzzy and a future fuzzy time intervals, respectively, in which the state $x \in X$ of the system is measured.

The family of fuzzy sets $\{E_{\tilde{Q}_\tau}(\cdot, \tau)\}_{\tau \geq 0}$ from $\overline{B}$ is called the trajectory of a future fuzzy process, and the family of fuzzy sets $\{E_{\tilde{Q}^*}(\cdot, \tau)\}_{\tau \geq 0}$ from $\overline{B}$ is called the trajectory of a current fuzzy process. The family of pairs of fuzzy sets $\{E_{\tilde{Q}_\tau}(\cdot, \tau), E_{\tilde{Q}^*}(\cdot, \tau)\}_{\tau \geq 0}$ is called the trajectory of an extremal fuzzy process $(\tilde{Q}_\tau, \tilde{Q}^*)$.

Let $\tilde{R}_\tau \subset X \times T \times Y$ be some lower measurable fuzzy relation $(\tilde{R}_\tau \in B \otimes B_{T^*} \otimes B_Y)$ describing expert knowledge reflections of fuzzy states of the system on the output values of the system in future fuzzy time intervals, and $\tilde{R}^* \subset X \times T \times Y$ be some upper measurable fuzzy relation $(\tilde{R}^* \in B \otimes B_{T^*} \otimes B_Y)$ describing expert knowledge reflections of fuzzy states of the system on the output values of the system in current fuzzy time intervals.

**Definition 2**

a) A lower measurable relation $\tilde{R}_\tau \in B \otimes B_{T^*} \otimes B_Y$ is called a future fuzzy process of expert knowledge reflection of states of the system in future fuzzy time intervals.

b) An upper measurable relation $(\tilde{R}^* \in B \otimes B_{T^*} \otimes B_Y)$ is called a current fuzzy process of expert knowledge reflection of states of the system in current fuzzy time intervals.

c) A pair $(\tilde{R}_\tau, \tilde{R}^*)$ is called an extremal fuzzy process of expert knowledge reflection of states of the system in extremal fuzzy time intervals.

Let $\tilde{R}_\tau \in (B \otimes B_{T^*}) \otimes (B_U \otimes B_{T^*}) \otimes \overline{B}$ be some lower measurable fuzzy relation in the Cartesian product $(X \times T) \times (U \times T) \times X$, which describes system state transformations in time with control taken into account:

$(X \times T) \times (U \times T) \rightarrow X$.

This relation is a future fuzzy transition operator describing the dynamics of the system or, in other words, system state transformations in future fuzzy time intervals.

Let $\tilde{R}^* \in (\overline{B} \otimes B_{T^*}) \otimes (B_U \otimes B_{T^*}) \otimes \overline{B}$ be some upper measurable fuzzy relation in the Cartesian product $(X \times T) \times (U \times T) \times X$, which describes system state transformations in time with control taken into account:

$(X \times T) \times (U \times T) \rightarrow X$.

This relation is a current fuzzy transition operator describing the dynamics of the system or, in other words, system state transformations in current fuzzy time intervals.
We call \( \tilde{\rho} \), the fuzzy lower transition operator describing the system state dynamics, and \( \tilde{\rho}^* \) the fuzzy upper transition operator describing the system state dynamics. The pair \( (\tilde{\rho}, \tilde{\rho}^*) \) is called the transition operator describing the system state dynamics in extremal fuzzy time intervals.

Let \( \tilde{u}^* \subset U \times T \) be some upper measurable fuzzy binary relation from \( \tilde{B}_T \otimes \tilde{B}_T^* \), which describes the action of external factors (controls) on the system in future fuzzy time intervals, and \( \tilde{u}_s \subset U \times T \) be some lower measurable fuzzy binary relation from \( \tilde{B}_U \otimes \tilde{B}_T^* \), which describes the action of external factors (controls) on the system in current fuzzy time intervals.

**Definition 3**

a) A fuzzy binary relation \( \tilde{u}^* \in \tilde{B} \otimes \tilde{B}_T^* \) is called a current fuzzy control process.

b) A fuzzy binary relation \( \tilde{u}_s \in \tilde{B} \otimes \tilde{B}_T^* \) is called a future fuzzy control process.

c) A pair \( (\tilde{u}_s, \tilde{u}^*) \) is called an extremal fuzzy control process.

d) The train

\[
\{ X, U, T, Y, (\tilde{\rho}, \tilde{\rho}^*), (\tilde{Q}_s, \tilde{Q}^*), (\tilde{R}_s, \tilde{R}^*) \}
\]

is called the extremal fuzzy continuous dynamic system (EFCDS) describing the state dynamics of the system in extremal fuzzy time intervals.

In the sequel we will consider the case with \( Y \equiv X \).

It is obvious that the EFCDS (3) describes the state dynamics of the system undergoing transformation with fuzzy uncertainty produced by observations at fuzzy time, while the extremality is due to the “measurement” of fuzzy states of the system in current and future fuzzy time intervals.

### 3 Continuous Extremal Controllable Fuzzy Process

**Definition 4**

a) A family \( \{ \tilde{r}_\tau^* \}_{\tau \geq 0} \) of monotonically increasing sequences of upper fuzzy time intervals, i.e.,

\[
\forall \tau_2 > \tau_1 \geq 0, \quad \tilde{r}_{\tau_1}^* \leq \tilde{r}_{\tau_2}^*
\]
is called a process of current fuzzy time intervals.

b) A family \( \{ \tilde{r}_\tau^* \}_{\tau \geq 0} \) of monotonically decreasing sequences of upper fuzzy time intervals, i.e.,

\[
\forall \tau_2 > \tau_1 \geq 0, \quad \tilde{r}_{\tau_1}^* \geq \tilde{r}_{\tau_2}^*
\]
is called a process of future fuzzy time intervals.

c) A pair of processes of future and current fuzzy time intervals \( \{ \tilde{r}_\tau, \tilde{r}_\tau^* \}_{\tau \geq 0} \) is called a process of extremal fuzzy time intervals.

A relation between the spaces \( (X, \tilde{B}, g) \) and \( (T, \tilde{B}_T, \tilde{B}_T^*, g_T, g_T^*) \) and their extensions through conditional measures can be represented as follows:

\[
\forall r_s \in \tilde{B}_T^*, r^* \in \tilde{B}_T^*, \tilde{r}_s \in \tilde{B}_T, \tilde{r}^* \in \tilde{B}_T^* \ [14]
\]

\[
g_T(r_s) = \int_X g_N(r_s \mid x) \circ g(\cdot),
\]

\[
g_T^*(r^*) = \int_X g_N^*(r^* \mid x) \circ g(\cdot),
\]

\[
\tilde{g}_T(r_s) = \int_X \tilde{g}_N(r_s \mid x) \circ g(\cdot),
\]

\[
\tilde{g}_T^*(r^*) = \int_X \tilde{g}_N^*(r^* \mid x) \circ g(\cdot),
\]

where \( \tilde{g} \) is a symbol of a Sugeno integral ([5], [12]).

By the definition of \( \tilde{g}_N(\cdot \mid x) \) and \( \tilde{g}_N^*(\cdot \mid x) \), for any lower and upper fuzzy time intervals \( \tilde{r}_s \in \tilde{B}_T^* \) and \( \tilde{r}^* \in \tilde{B}_T^* \) there exist \( \tilde{B} \)-measurable sets \( \tilde{A}_{\tilde{r}_s} \subset \tilde{B} \), \( \tilde{A}_{\tilde{r}^*} \subset \tilde{B} \) such that \( \forall x \in X \ [14] \)

\[
\mu_{\tilde{A}_{\tilde{r}_s}}(x) = \tilde{g}_N(\tilde{r}_s \mid x), \quad \mu_{\tilde{A}_{\tilde{r}^*}}(x) = \tilde{g}_N^*(\tilde{r}^* \mid x). \]

**Definition 5**

The fuzzy sets \( \tilde{A}_{\tilde{r}_s} \) and \( \tilde{A}_{\tilde{r}^*} \) are measurable from the extended measurable space of system states are called the expert reflections of an extremal fuzzy dynamic systems states in the extremal fuzzy time intervals \( \tilde{r}_s, \tilde{r}^* \) with respect to extended extremal conditional fuzzy measures \( \tilde{g}_N(\cdot \mid x) \) and \( \tilde{g}_N^*(\cdot \mid x) \).

Let us formulate a theorem that describes the ergodicity of an expert reflection process in an ergodic process of extremal fuzzy time intervals [14]:

**Theorem 1** An ergodic process \( (\tilde{r}_\tau, \tilde{r}_\tau^*)_{\tau \geq 0} \) of extremal fuzzy time intervals on the measurable space of states of the system \( (X, \tilde{B}) \) induces an ergodic expert reflection process \( (\tilde{R}_\tau, \tilde{R}_\tau^*) \equiv (\tilde{A}_{\tilde{r}_\tau}, \tilde{A}_{\tilde{r}_\tau^*})_{\tau \geq 0} \).

### 4 The Fuzzy Dynamic Programming Problem

We consider the optimization problem of EFCDS when the model of the continuous extremal controllable fuzzy process is described by the system of fuzzy
Then the complement $\tilde{Q}_{\ast}$ is a fuzzy extremal process describing the system state dynamics; $(\tilde{\mathbb{R}}_{\ast}, \tilde{\mathbb{R}}^\ast)$ is an extremal fuzzy process of expert knowledge reflections in extremal fuzzy time intervals (the expert reflections on the states of EFCDS in the extremal fuzzy time intervals); $(\tilde{p}_{\ast}, \tilde{p}^\ast)$ is the transition operator of the EFCDS states; on right-hand sides of Sugeno extended lower states; on right-hand sides of Sugeno extended lower time intervals (the expert reflections on future loss $\tilde{u}_{\ast} \in \tilde{B}_{U} \otimes \tilde{B}_{T^\ast}$ on a future fuzzy time interval $\tilde{r}_{\ast} \in \tilde{B}_{T^\ast}$.

We have thus defined, on $U$, an extremal fuzzy "gain-loss" process $(I_u^\ast, J_u^\ast)$. Further, for model (6) we will consider, in terms of (9) and (10), the problem of formation of an optimal control (in the sense of minimization of the future loss and maximization of the current gain) of an extremal process: $\forall (u, t) \in U \times T$

\begin{equation}
\int_{T}^{\infty} \mathbf{P}^{K}_{\tilde{u}_{\ast}}(u, t) \circ \tilde{g}_{\tilde{E}_{\tilde{u}_{\ast}}}(\cdot) \Rightarrow \max_{\tilde{u}_{\ast}}
\end{equation}

\begin{equation}
\int_{T}^{\infty} \mathbf{q}^{K}_{\tilde{u}_{\ast}}(u, t) \circ \tilde{g}_{\tilde{E}_{\tilde{u}_{\ast}}}(\cdot) \Rightarrow \min_{\tilde{u}_{\ast}}.
\end{equation}

Functional equations by means of which we can define an extremal fuzzy optimal control in the sense of extremalization of criteria (11) can be written in the following form, $\forall (u, t') \in U \times [\tau_0, T]$:

\begin{equation}
\begin{cases}
\tilde{J}_{\tilde{u}_{\ast}}(u, t') = \text{max} \sum_{\tilde{u}_{\ast} \in \tilde{B}_{U} \otimes \tilde{B}_{T^\ast}} J_{\tilde{u}_{\ast}}(u, t') \\
I_{\tilde{u}_{\ast}}(u, t') = \text{max} \sum_{\tilde{u}_{\ast} \in \tilde{B}_{U} \otimes \tilde{B}_{T^\ast}} I_{\tilde{u}_{\ast}}(u, t')
\end{cases}
\end{equation}

with the initial control conditions

$\tilde{E}_{\tilde{u}_{\ast}}(\cdot, \tau_0) = \tilde{u}_{0 \ast} \in \tilde{B}_{U}$, $\tilde{E}_{\tilde{u}_{\ast}}(\cdot, \tau_0) \equiv \tilde{u}_{0 \ast} \in \tilde{B}_{U}$

and the EFCDS initial states $\tilde{E}_{\tilde{Q}_{\ast}}(\cdot, \tau_0)$ and $\tilde{E}_{\tilde{Q}_{\ast}}(\cdot, \tau_0)$.

\textbf{Definition 7} An extremal fuzzy control process $(\tilde{u}_{\ast}, \tilde{u}^\ast)$, $\tau_0 \leq \tau' \leq \tau$, with the initial conditions (13) is called an optimal for EFCDS (6) in the sense of Bellman’s optimality principle if criterion (12) is satisfied.

The following theorems which gives the optimality condition (an analogue of Bellman’s equation [1], [2]) are valid.

\textbf{Theorem 2} Let a EFCDS be described by system (6). Then an extremal fuzzy control process $(\tilde{u}_{\ast}, \tilde{u}^\ast)$,
\( \tau_0 \leq \tau' \leq \tau \), is optimal in the sense of criterion (12) if and only if the following inequalities are fulfilled:

\[
\forall (u, \tau') \in U \times [\tau_0, \tau] \\
J_{\tilde{u}_*}(u, \tau') \leq \left( \int_{K} \mu_L(v, u) \circ \tilde{g}_K(\cdot) \right) \land \mu_{E_{\tilde{g}_*}}(\cdot, \tau_0)(u), \\
I_{\tilde{u}_*}(u, \tau') \geq \left( \int_{K} \mu_L(v, u) \circ \tilde{g}_K(\cdot) \right) \lor \mu_{E_{\tilde{g}_*}}(\cdot, \tau_0)(u);
\]

(14)

**Theorem 3** An extremal fuzzy optimal control process \((\hat{u}_*, \hat{u}^*)\) for the EFCDS (6) in the sense of criterion (12) not depending on a EFCDS state can be defined by the following system of fuzzy-integral equations:

\[
\forall (u, \tau') \in U \times [\tau_0, \tau] \\
\mu_{\hat{u}_*}(u, \tau') = \mu_{\hat{u}_*}(u, \tau_0) \land \left( \int_{K} \mu_L(v, u) \circ \tilde{g}_K(\cdot) \right) \land \hat{g}_{E_{\tilde{g}_*}}(\cdot, \Delta(\tau_0, \tau'))(T), \\
\mu_{\hat{u}^*}(u, \tau') = \mu_{\hat{u}^*}(u, \tau_0) \lor \left( \int_{K} \mu_L(v, u) \circ \tilde{g}_K(\cdot) \right) \lor \hat{g}_{E_{\tilde{g}_*}}(\cdot, \Delta(\tau_0, \tau'))(T). \\
\]

(15)

Applications and examples of the existence of a fuzzy optimal control see in [16], [18], [20], [21].

## 5 Identification Method and Algorithm for the Model of Continuous Extremal Fuzzy Process (EFP)

In this section we consider some problems of the identification of the model of the extremal fuzzy continuous dynamic system (EFCDS) that we have presented in Section 4 (equations (6)), where the control parameter \(U\) is omitted.

The basic approaches to the identification of fuzzy process models that have been developed to this day [3], [6], [7], [9]–[11], [15], and other works can be divided into two groups – analytical and algorithmic – both of which are oriented to a fuzzy process model written in terms of fuzzy compositional or integro-differential equations or their modifications. Various analytical methods and algorithms were used in order to identify such models, i.e., the corresponding relation of spaces of inputs and outputs of fuzzy dynamic systems. These methods mainly imply the construction of some set-theoretic operation that is inverse to the composition operation and requires a subsequent smoothing of the results. In some works fuzzy models of regression type were identified by means of analytical regularization methods that allowed one to obtain numerical estimators of model coefficients. In this paper, a new approach is proposed to the identification of EFCDS models.

Let the model of a continuous extremal fuzzy process be described by the system of fuzzy-integral equations (6), where the parameter of a control is omitted. Assume that as a result of observations on the extremal fuzzy process describing changes in the system state dynamics with possibilistic uncertainty we have \(N\) measurements of fuzzy states of input pairs \(\langle \hat{A}_{0*\tau_i}, \hat{A}^*_{0*\tau_i} \rangle\) and output pairs \(\langle \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_1), \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i) \rangle\), \(i = 1, 2, \ldots, N\), i.e., data on “true” states of EFCDS input-output states measured respectively in extremal fuzzy time intervals \((\hat{r}_{\tau_i*}, \hat{r}_{\tau_i}^*)\). \(i = 1, 2, \ldots, N\), where \(\tau_1 < \tau_2 < \cdots < \tau_N\). It is obvious that in the measurement process realization, true and model data on input pairs coincide:

\[
\langle \hat{A}_{0*\tau_i*}, \hat{A}^*_{0*\tau_i} \rangle = \langle \hat{A}_{0*\tau_i}, \hat{A}^*_{0*\tau_i} \rangle, \quad i = 1, 2, \ldots, N, \quad (16)
\]

while true and model data on output pairs may differ because true EFP data are possibilistic-experimental and model data are defined by (6):

\[
\langle \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i), \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i) \rangle \neq \langle \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i), \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i) \rangle \\
\]

\[
\quad i = 1, 2, \ldots, N. \quad (17)
\]

Clearly, to obtain an acceptable solution of the identification problem, from the standpoint of application it seems most logical to use a regularization condition that provides a “set covering” solution or, in other words, allows one to obtain such estimators of transition operator that guarantees the fulfillment of the following condition: for any \(N\) input data \(\langle \hat{A}_{0*\tau_i*}, \hat{A}^*_{0*\tau_i} \rangle, \quad i = 1, \ldots, N\), we have

\[
\langle \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i), \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i) \rangle \supseteq \langle \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i), \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i) \rangle \quad \iff \quad \langle \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i), \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i) \rangle \quad \iff \\
\hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i) \supseteq \hat{E}_{\tilde{Q}_*}, (\cdot, \tau_i), \quad i = 1, \ldots, N. \quad (18)
\]

As a rule the relation \(\hat{E}_{\tilde{Q}_*} \lesssim \hat{E}_{\tilde{Q}_*}\) and therefore condition (18) are understood in terms of finding a model EFP \(\langle \tilde{Q}_*, \tilde{Q}_*^* \rangle\), where, at “experimental points”, model fuzzy extremal states are less vague than experimental data.

In view of the regularization condition (18), the fuzzy-integral model (6) of a continuous EFP can be identified by means of following theorem [15].
Theorem 4 The EFP fuzzy-integral model (6) based on N “input ⇒ output” measurements
\[
\langle \tilde{A}_{0, \tau_i}, \tilde{A}^*_0 \rangle \Rightarrow (\tilde{E}^*_{\tilde{Q}_s} (\cdot, \tau_i), \tilde{E}^*_{\tilde{Q}^*} (\cdot, \tau_i)), \quad i = 1, \ldots, N,
\]

is defined, in view of the regularizarion condition (18), in extremal fuzzy time intervals \((\tilde{r}_{\tau^*_i}, \tilde{r}^0_{\tau^*_i})\), \(i = 1, 2, \ldots, N\), by the following transition operator \((\tilde{\rho}_s, \tilde{\rho}^*):\) for fixed \(x \in X\) and \(\forall (x', \tau) \in X \times T\)
\[
\mu_{\tilde{E}_{\tilde{Q}_s}}(x, \tau)(x', \tau) = \bigwedge_{i=1}^N \mu_{\tilde{Q}_s}^i (x, \tau_i) \land \mu_{\tilde{H}_i}(x', \tau), \\
\mu_{\tilde{E}_{\tilde{Q}^*}}(x, \tau)(x', \tau) = \bigvee_{i=1}^N \mu_{\tilde{Q}^*}^i (x, \tau_i) \lor \mu_{\tilde{H}_i}(x', \tau),
\]

(20)

where \(\tilde{H}_i \in \tilde{B} \otimes \tilde{B}_{T^*_i}\) is a lower and \(\tilde{H}^*_i \in \tilde{B} \otimes \tilde{B}^*_{T^*_i}\) an upper binary relation which for the \(i\)-th measurement are defined as follows:
\[
\begin{align*}
\tilde{H}_i = & \tilde{E}_{\tilde{Q}_s} \otimes \tilde{H}_{i}^* (\tilde{E}_{\tilde{Q}_s}), \\
\tilde{H}_i^* = & \tilde{E}_{\tilde{Q}^*} \otimes \tilde{H}_{i}^* (\tilde{E}_{\tilde{Q}^*}), \\
\tilde{E}_{i} = & \{ x'' \in X | x'' \in \tilde{Q}_s, x'' \leq \tilde{Q}_s (\cdot) (x) \}, \\
g(\tilde{E}_{i}) = & \mu_{\tilde{Q}_s}^i (\cdot, \tau_i)(x), \\
\tilde{E}_i^* = & \{ x'' \in X | x'' \in \tilde{Q}^*_s, x'' \geq \tilde{Q}^*_s (\cdot) (x) \}, \\
g^*(\tilde{E}_{i}^*) = & \mu_{\tilde{Q}^*_s}^i (\cdot, \tau_i)(x), \\
\tilde{g}_{ts}(\tilde{H}^*_i) (\tilde{E}_{i} \times x') \leq \tilde{E}_{\tilde{Q}_s} (\cdot, \tau_i)(x), \\
\tilde{g}_{ts}(\tilde{H}^*_i) (\tilde{E}_{i}^* \times x') \geq \tilde{E}_{\tilde{Q}^*_s} (\cdot, \tau_i)(x),
\end{align*}
\]

where \(\tilde{H}^*_i, \tilde{H}_{i}^* \) are extended mappings of the Galois correspondence.

Remark 1 If \((\tilde{\rho}_s, \tilde{\rho}^*)\) is some solution of the problem of transition operator identification in the conditions of Theorem 4, then any transition operator \((\tilde{\rho}_s^*, \tilde{\rho}^*_s)\), \(\tilde{\rho}_s^* \subseteq \tilde{\rho}_s, \tilde{\rho}^*_s \supseteq \tilde{\rho}^*\) is also a solution of this problem, i.e., it satisfies the regularization conditions (18). This means that, using data on the EFP, the solution \((\tilde{\rho}_s, \tilde{\rho}^*)_s\) is unique in the sense of embeddedness of fuzzy relations.

Remark 2 Denote by \((\tilde{\rho}_s (\tilde{H}_s), \tilde{\rho}^* (\tilde{H}^*))\) a solution of the problem of transition operator’s identification in the conditions of Theorem 4 for admissible extremal fuzzy binary relations \(H_s = (\tilde{H}_1, \tilde{H}_2, \ldots, \tilde{H}_N) \in (\tilde{B} \otimes \tilde{B}_{T^*_i})^N\), \(\tilde{H}^* = (\tilde{H}^*_1, \tilde{H}^*_2, \ldots, \tilde{H}^*_N) \in (\tilde{B} \otimes \tilde{B}^*_{T^*_i})^N\). Then it is obvious that if \(\tilde{\rho}_s \equiv \bigcap_{j=1}^N \tilde{\rho}_j (\tilde{H}_j) \in \tilde{B} \otimes \tilde{B}_{T^*_i}\) and if \(\tilde{\rho}^* \equiv \bigcup_{j=1}^N \tilde{\rho}^*_j (\tilde{H}^*_j) \in \tilde{B} \otimes \tilde{B}^*_{T^*_i}\), then \((\tilde{\rho}_s, \tilde{\rho}^*)\) is also a solution of the identification problem for all admissible \((\tilde{H}_s, \tilde{H}^*)\).

Remark 3 The problem of finding the existence conditions for \((\tilde{\rho}_s, \tilde{\rho}^*)\) is the subject of our future research. For the finite case of our future research \((X\) is finite) this problem maybe approximately solved by the genetic algorithms.

Theorem 4 enables us to define an identification algorithm of the transition operator \((\tilde{\rho}_s, \tilde{\rho}^*)\) of a continuous EFP.

Algorithm 1. 1) For fixed \(x \in X\) \((X\) is a finite set), monotone sequences of \(\alpha\)-levels are defined for all \(N\) measurements separately for \(\tilde{Q}_s\) and separately for \(\tilde{Q}^*_s\). Let
\[
\mu_{s*} \equiv \mu_{\tilde{Q}_s^*} (\cdot, \tau_i)(x), \quad \mu^*_i \equiv \mu_{\tilde{Q}^*_s} (\cdot, \tau_i)(x),
\]

so that
\[
\mu_1^* \leq \mu_2^* \leq \cdots \mu_N^*, \quad \mu^*_1 \geq \mu^*_2 \geq \cdots \geq \mu^*_N. \quad (22)
\]

2) For \(j = 1, 2, \ldots, N\) we define subsets of cuttings of \(\tilde{E}_{j*} \) and \(\tilde{E}_{j}^* \) and the sets \(\tilde{H}_{j*} \), \(\tilde{H}_{j}^* \) by means of conditions (21).

3) Solutions are constructed for the \(j\)-th step \((j = 1, 2, \ldots, N)\):
\[
\mu_{1*j} (x, x', t) = \mu_{j*} \land \mu_{\tilde{H}_{j*}} (x', t), \\
\mu_{j*} (x, x', t) = \mu_{j*} \lor \mu_{\tilde{H}_{j}^*} (x', t).
\]

Steps 2) and 3) are repeated for all \(j = 1, 2, \ldots, N\).

4) For fixed \(x \in X\) and \(\forall (x', t) \in X \times T\), estimates \((\tilde{\rho}_s, \tilde{\rho}^*)\) are constructed:
\[
\mu_{1*j} (x, x', t) = \bigwedge_{j=1}^N \mu_{1*j} (x, x', t), \\
\mu_{j*} (x, x', t) = \bigvee_{j=1}^N \mu_{j*} (x, x', t).
\]

Steps 1)–4) are repeated for all \(x \in X\). This completes the numerical restoration of the transition operator \((\tilde{\rho}_s, \tilde{\rho}^*)\), using the regularization condition (18).

The above algorithm makes it possible to obtain approximated solutions as estimators and thus to construct fuzzy-integral models (6) of an EFP.
most difficult point in the realization of this algorithm is the definition of extremal fuzzy time intervals \( \Gamma_{\tilde{H}_r}^\tau(\tilde{E}_{j,s}) \in \tilde{B}_T \) and \( \Gamma_{\tilde{H}_r}^\tau(\tilde{E}_{k,s}) \in \tilde{B}_T' \) for which conditions (21) are fulfilled. As mentioned in Remark 2, the solution of this problem is not unique and therefore in this case too we have to introduce regularization conditions for finite \( X \). In doing so, we use the concrete parametric representations of extremal fuzzy time intervals.

Applications and examples of the proposed algorithm of EFP model identification by means of Theorem 4 are presented in [15].

6 Conclusions

Using the results obtained in [21]–[21] we shortly described the issues of controllable extremal process modeling. The following problems have been already researched and solved:

a) the basic properties of Sugeno’s type extremal fuzzy measure and several variants of its representations are considered [12];

b) the notions of extremal fuzzy time moments and intervals are introduced and their monotone algebraic structures are defined. The dualization of a time structure forms the most important part of the fuzzy instrument of modeling and optimization of extremal fuzzy continuous dynamic systems [12];

c) we introduce the notion of an EFCDS with fuzzy uncertainty, the source of which is “fuzzy measurement” (“expert reflections” on the states of EFCDS) of the system state in the so-called current and future fuzzy time intervals. The general EFCDS model is described [14];

d) the notion of processes of expert reflection and description of the EFCDS state change dynamics are introduced. With the aid of the conditional extremal expert reflection measures \( g_{ts}(\cdot \mid x) \) and \( g_{ts}(\cdot \mid x) \), the extremal fuzzy reflection process \((\tilde{\mathbb{R}}_x, \tilde{\mathbb{R}}^*)\) connects the fuzzy time interval measurement process \((\tilde{r}_{ts}, \tilde{r}^*_s)_{t \geq 0}\) with the space of measurable states of the system with fuzzy distribution \((X, B, g)\), while the EFCDS state description process \((\tilde{Q}_s, \tilde{Q}^*_s)\) is defined through the extremal fuzzy reflection process \((\tilde{\mathbb{R}}_x, \tilde{\mathbb{R}}^*)\), using the extended upper and lower Sugeno integrals that are considered as extremal operators describing the EFCDS state dynamics [14];

e) consideration is given to the continuous and discrete case of extremal fuzzy processes. A question of the ergodicity of extremal fuzzy processes is studied, which allows one to obtain a sufficient condition for the process \((\tilde{Q}_s, \tilde{Q}^*_s)\) to be ergodic [14];

f) methods and algorithms have been developed for identifying the transition operator of an EFCDS by using the fuzzy-integral model (6) and the information on realized \( N \) input-output pairs (19) [15];

g) the restoration of the transition operator by experimental data in model (6) is a nonregular problem. In order to obtain a unique (in a certain sense) quasioptimal solution of the identification problem, the following regularization condition is introduced for continuous EFPs this is the principle of covering of the experimental data (18) by the corresponding modeled extremal data [15];

h) the respective algorithms of identification of \((\tilde{\rho}^*, \tilde{\rho}^*_s)\) have been developed. The results obtained are illustrated by the examples with a finite set of EFCDS states. A good agreement between the estimates obtained by the proposed method and experimental data is observed [15];

i) we have introduced the notion of the weakly structurable controllable dynamic system (WSCDS) in case of fuzzy control action, where the source of uncertainty is expert reflections (expert measurements) of the system states in monotonically increasing fuzzy time intervals [14];

j) the problem of (6) WSCDS optimal control have been studied in [16];

k) the problems of estimation (filtration) of (6) WSCDS states have been studied in [17].

The following problems need further research:

1) the problem of estimation of pessimistic-optimistic indices of ergodicity for each problem of extremal fuzzy processes fuzzy-modeling (identification, fuzzy-optimal control, fuzzy-filtration);

2) the quantitative-basic analysis of adaptation as object of WSDS control in the environment of anomalous and extremal processes;

3) construction of possibilistic-objective simulation algorithms for anomalous and extremal processes based on constructed models;

4) creation of adaptation scenarios in the environment of anomalous and extremal processes using expert-possibilistic theory;

5) development of software for universal library implementing the WSDS structure and decision support methods; Creation of decision-support systems for real applications.

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