Abstract: We study the periodic solutions of a Poisson-gradient PDEs system with bounded non-linearity. Section 1 introduces the basic spaces and functionals. Section 2 studies the weak differential of a function and establishes an inequality. Section 3 formulates some conditions under which the action functional is continuously differentiable. Section 4 analyzes the Poisson-gradient systems and some conditions that ensure periodic solutions.

Key–Words: variational methods, elliptic systems, periodic solutions, weak derivatives, potentials.

1 Introduction

We consider the point \( T = (T^1, ..., T^p) \) and the parallelepiped \( T_0 = [0, T^1] \times ... \times [0, T^p] \) in \( \mathbb{R}^p \). We denote by \( W_{T,1}^{1,2} \) the Sobolev space of the functions \( u \in L^2 [T_0, \mathbb{R}^n] \) which have the weak derivative \( \frac{\partial u}{\partial t} \in L^2 (T_0, \mathbb{R}^n) \). The index \( T \) from the notation \( W_{T,1}^{1,2} \) comes from the fact that the weak derivatives are defined using the space \( C^\infty_T \) of all indefinitely differentiable multiple \( T \)-periodic functions from \( \mathbb{R}^p \) into \( \mathbb{R}^n \). We denote by \( H^T_1 \) the Hilbert space \( W_{T,1}^{1,2} \). The norm used in \( H^T_1 \) is the one induced by the scalar product

\[
\langle u, v \rangle = \int_{T_0} \left( \delta_{ij} u^i (t) v^j (t) \right) + \delta_{ij} \delta^{\alpha \beta} \frac{\partial u^i}{\partial t} (t) \frac{\partial v^j}{\partial t} (t) dt^1 \wedge ... \wedge dt^p.
\]

These are induced by the scalar product (Riemanian metric)

\[
G = \begin{pmatrix}
\delta_{ij} & 0 \\
0 & \delta^{\alpha \beta} \delta_{ij}
\end{pmatrix}
\]

on \( \mathbb{R}^{n+p} \) (multiphase space) and its associated Euclidean norm. We shall also use the scalar product \( \langle u, v \rangle = \delta_{ij} u^i v^j \) and the norm \( |u| = \sqrt{\delta_{ij} u^i u^j} \) simultaneously, from the Euclidean space \( \mathbb{R}^n \).

Let \( t = (t^1, ..., t^p) \) be a generic point in \( \mathbb{R}^p \). Then the opposite faces of the parallelepiped \( T_0 \) can be described by the equations

\[
S^+_i : t^i = 0, S^-_i : t^i = T^i
\]

for each \( i = 1, ..., p \). We shall study the minimum of the action

\[
\varphi (u) = \int_{T_0} L \left( t, u (t), \frac{\partial u}{\partial t} (t) \right) dt^1 \wedge ... \wedge dt^p,
\]

\[
L \left( t, u (t), \frac{\partial u}{\partial t} (t) \right) = \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + F (t, u (t))
\]

on the space \( H^T_1 \), considering that the potential function \( F \) has the property of bounded non-linearity. We use the method of the minimizing sequences and the coercitivity condition

\[
\int_{T_0} F \left( t, u (t) \right) dt^1 \wedge ... \wedge dt^p \to \infty
\]

when \( |u| \to \infty \). The extremals of the action \( \varphi \) verifies the Euler-Lagrange equations with the boundary conditions

\[
\left. u \right|_{S^+_i} = \left. u \right|_{S^-_i} \quad \left. \frac{\partial u}{\partial t} \right|_{S^+_i} = \left. \frac{\partial u}{\partial t} \right|_{S^-_i}, \quad i = 1, ..., p.
\]

Due to the particularity of the Lagrangian \( L \), the Euler-Lagrange equations reduce to a PDEs system of the Poisson-gradient type

\[
\Delta u (t) = \nabla F (t, u (t)).
\]

The aim of this paper is to discuss the existence of solutions of this PDEs system with suitable boundary conditions. More precisely, we extend the theory in [2] from single-time to multi-time field theory, developing the ideas in the papers [6], [7], [9].
In this way we find positive answers for the existence of multi-periodical solutions of Euler-Lagrange equations that are Poisson-gradient PDEs with bounded non-linearity. The original results, announced in [13], can be applied to the multi-time geometric dynamics ([5], [8], [10]-[12]).

2 On the weak differential of a function

We consider $C^\infty$ the space of the indefinitely differentiable functions multiple periodical with the period $T = (T^1, ..., T^p)$, defined on $R^p$ taking values in $R^n$.

We know that $C^\infty_T \subset W^{1,2}_T$. We establish some conditions satisfied by a function $u \in L^1 [T_0, R^n]$ which has a weak differential.

**Theorem 1.** Let $u, v_\alpha \in L^1 [T_0, R^n], \alpha = 1, ..., p$, such that $v_\alpha dt^\alpha = (v_1 dt^1, ..., v_p dt^p)$ is an integrable vector form. We consider $\tilde{OT}$ an arbitrary curve from $T_0$, having the endings at $O = (0, ..., 0)$ and $T = (T^1, ..., T^p)$.

If

$$\int_{\tilde{OT}} (u, df) = -\int_{\tilde{OT}} (v_\alpha dt^\alpha, f),$$

for any $f \in C^\infty$, then $\int_{\tilde{OT}} v_\alpha dt^\alpha = 0$ and it exists $c \in R^n$ such that $u(t) = \int_{\tilde{OT}} v_\alpha ds^\alpha + c$. Also $u(0) = u(T)$.

**Proof.** We choose $f = e^i = (0, ..., 0, 1, 0, ..., 0)$, with the value 1 on the position $i$. From the relation (1) we have $0 = -\int_{\tilde{OT}} v_\alpha dt^\alpha$ and hence $\int_{\tilde{OT}} v_\alpha dt^\alpha = 0$.

We define $w \in C (T_0, R^n)$ by $w(t) = \int_{\tilde{OT}} v_\alpha ds^\alpha, t \in \tilde{OT}$. By Fubini Theorem, the function $w$ satisfies the relation

$$\int_{\tilde{OT}} w(t, df) = \int_{\tilde{OT}} (\int_{\tilde{OT}} v_\alpha ds^\alpha, df)$$

$$= \int_{\tilde{OT}} (\int_{\tilde{OT}} v_\alpha df^\alpha) ds^\alpha = \int_{\tilde{OT}} (v_\alpha, f(T) - f(s)) ds^\alpha$$

$$= -\int_{\tilde{OT}} (v_\alpha, f(s)) ds^\alpha = \int_{\tilde{OT}} (u, df).$$

This means that

$$\int_{\tilde{OT}} (u - w, df) = 0.$$  

We consider now $\gamma : [a, b] \to T_0, \gamma (\xi) = (t^1(\xi), ..., t^p(\xi)), \gamma (a) = O, \gamma (b) = T$, a parametrization of the curve $\tilde{OT}$. The equality (2) becomes

$$\int_a^b (u(t(\xi)) - w(t(\xi))),$$

$$\left(\frac{\partial f^1}{\partial t^1} dt^1, ..., \frac{\partial f^p}{\partial t^p} dt^p\right) d\xi = 0,$$

for any $f \in C^\infty_T$. We will particularize for the function sequences

$$f_j^{(k)} (t) = \begin{cases} \cos & k \in N \setminus \{0\}, \\ \sin & 1 \leq j \leq n \end{cases}$$

and we observe that (see the Fourier series theory) $u(t) - w(t) = c$, $c \in R^n$ almost everywhere in $T_0$ (the constant is the only function orthogonal to the previous sequences). By replacing $w(t)$, we find that $u(t) = \int_{\tilde{OT}} v_\alpha ds^\alpha + c$ for any $t \in \tilde{OT}$. The function $u$ satisfies $u(0) = c$ and $u(T) = \int_{\tilde{OT}} v_\alpha ds^\alpha + c = c$, so $u(0) = u(T)$.

On the other side, the relation $u(t) - u(\tau) = \int_{\tau T} v_\alpha ds^\alpha$ implies that $u(t) = \int_{\tau T} v_\alpha ds^\alpha + u(\tau)$. The 1-form $v_\alpha dt^\alpha$ is called weak differential of the function $u$. By a Fourier series argument, the weak differential, if it exists, is unique. The weak differential of $u$ will be denoted by $du$. The existence of $du$ implies $u(0) = u(T)$.

**Theorem 2.** If $u = (u^1, ..., u^n) \in L^2 [T_0, R^n]$, $|u(t)|^2 = \delta_{ij} u^i(t) u^j(t)$, then

$$\left| \int_{T_0} u(t) dt^1 \wedge ... \wedge dt^p \right| \leq (nT^1...T^p)^{\frac{1}{2}} \left( \int_{T_0} |u(t)|^2 dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}}.$$  

**Proof.** Successively we have the relations

$$\left| \int_{T_0} u(t) dt^1 \wedge ... \wedge dt^p \right| =$$

$$= \left| \int_{T_0} (u^1(t), ..., u^n(t)) dt^1 \wedge ... \wedge dt^p \right|$$

$$= \left| \left( \int_{T_0} u^1(t) dt^1 \wedge ... \wedge dt^p, ..., \int_{T_0} u^n(t) dt^1 \wedge ... \wedge dt^p \right) \right|$$

$$= \left( \left( \int_{T_0} u^1(t) dt^1 \wedge ... \wedge dt^p \right)^2 + ... + \left( \int_{T_0} u^1(t) dt^1 \wedge ... \wedge dt^p \right)^2 \right)^{\frac{1}{2}}.$$
\[ \leq \left| \int_{T_0} u^1(t) dt^1 \wedge ... \wedge dt^p \right| + ... + \left| \int_{T_0} u^n(t) dt^1 \wedge ... \wedge dt^p \right| \]
\[ \leq \int_{T_0} \left( |u^1(t)| + ... + |u^n(t)| \right) dt^1 \wedge ... \wedge dt^p \]
\[ = \int_{T_0} \left( \left( |u^1(t)| ,..., |u^n(t)|, (1,...,1) \right) dt^1 \wedge ... \wedge dt^p. \]

Using the Cauchy-Schwartz inequality, we obtain
\[ \left( \int_{T_0} |u(t)|^2 dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \leq \left( \int_{T_0} \lambda(t) dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}} \]
\[ = \left( nT^1 ... T^p \right)^{\frac{1}{2}} \left( \int_{T_0} |u(t)|^2 dt^1 \wedge ... \wedge dt^p \right)^{\frac{1}{2}}. \]

3 Continuously differentiable action

The next theorem establishes some conditions in which the action
\[ \varphi : W^{1,2}_T \to R, \varphi (u) = \int_{T_0} L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right) dt^1 \wedge ... \wedge dt^p \]
is continuously differentiable. In this way we extend the particular case \( p = 1 \), studied in [3, Theorem 1.4].

**Theorem 3.** We consider \( L : T_0 \times R^n \times R^{np} \to R, (t, x, y) \to L (t, x, y) \), a measurable function in \( t \) for any \( (x, y) \in R^n \times R^{np} \) and with the continuous partial derivatives in \( x \) and \( y \) for any \( t \in T_0 \). If there exist \( a \in C^1 (R^+, R^+) \) with the derivative \( a' \) bounded from above, \( b \in C (T_0, R^n) \) such that for any \( t \in T_0 \) and any \( (x, y) \in R^n \times R^{np} \) to have
\[ |L (t, x, y)| \leq a \left( |x| + |y|^2 \right) b(t), \]
\[ |\nabla_x L (t, x, y)| \leq a \left( |x| \right) b(t), \]
\[ |\nabla_y L (t, x, y)| \leq a \left( |y| \right) b(t), \]
then, the functional \( \varphi \) has continuous partial derivatives in \( W^{1,2}_T \) and its gradient derives from the formula
\[ \left( \nabla \varphi (u), v \right) = \int_{T_0} \left[ \left( \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), v(t) \right) + \left( \nabla_y L \left( t, u(t), \frac{\partial u}{\partial t} (t), \frac{\partial v}{\partial t} (t) \right) \right) \right] dt^1 \wedge ... \wedge dt^p. \]

**Proof.** It is enough to prove that \( \varphi \) has the derivative \( \varphi' (u) \in \left( W^{1,2}_T \right)^* \) given by the relation (4) and the function \( \varphi' : W^{1,2}_T \to \left( W^{1,2}_T \right)^* \), \( u \to \varphi' (u) \) is continuous. We consider \( u, v \in W^{1,2}_T, t \in T_0, \lambda \in [-1,1] \). We build the functions
\[ F (\lambda, t) = L \left( t, u(t) + \lambda v(t), \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right) \]
and
\[ \Psi (\lambda) = \int_{T_0} F (\lambda, t) dt^1 \wedge ... \wedge dt^p. \]
Because the derivative \( a' \) is bounded from above, exist \( M > 0 \) such that \( a' (|u| - a (0)) = \lambda \leq M. \) This means that \( a (|u|) \leq M |u| + a (0). \) On the other side
\[ \frac{\partial F}{\partial \lambda} (\lambda, t) = \left( \nabla_x L \left( t, u(t) + \lambda v(t), \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right), v(t) \right) + \left( \nabla_y L \left( t, u(t) + \lambda v(t), \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right), \frac{\partial v}{\partial t} (t) \right) \leq a (|u(t) + \lambda (v(t))|) \]
\[ b(t) |v(t)| + a \left( \left( \frac{\partial u}{\partial t} (t) + \lambda \frac{\partial v}{\partial t} (t) \right), b(t) \right) \frac{\partial v}{\partial t} (t) \]
\[ \leq b_0 \left( M (|u(t)| + |v(t)|) + a (0) \right) |v(t)| + \]
\[ b_0 \left( M \left( \left| \frac{\partial u}{\partial t} (t) \right| + \left| \frac{\partial v}{\partial t} (t) \right| \right) + a (0) \right) \frac{\partial v}{\partial t} (t), \]
where
\[ b_0 = \max_{t \in T_0} b(t). \]

Then, we have \( \frac{\partial F}{\partial \lambda} (\lambda, t) \leq d (t) \in L^1 (T_0, R^+). \)
Then Leibniz formula of differentiation under integral sign is applicable and
\[ \frac{\partial \Psi}{\partial \lambda} (0) = \int_{T_0} \frac{\partial F}{\partial \lambda} (0, t) dt^1 \wedge ... \wedge dt^p \]
\[ = \int_{T_0} \left[ \left( \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right), v(t) \right) + \left( \nabla_y L \left( t, u(t), \frac{\partial u}{\partial t} (t), \frac{\partial v}{\partial t} (t) \right) \right) \right] dt^1 \wedge ... \wedge dt^p. \]
Moreover,
\[ \left| \nabla_x L \left( t, u(t), \frac{\partial u}{\partial t} (t) \right) \right| \leq b_0 \left( M |u(t)| + a (0) \right) \]
\[ \in L^1 (T_0, R^+) \]
4 Poisson-gradient systems and their periodical solutions

4.1 Multi-time Euler-Lagrange equations

We consider the multi-time variable \( t = (t^1, ..., t^p) \in R^p \), the functions \( x^i : R^p \to R, (t^1, ..., t^p) \to x^i (t^1, ..., t^p), i = 1, ..., n \), and the partial velocities \( x^i_\alpha = \frac{\partial x^i}{\partial t^\alpha}, \alpha = 1, ..., p \). The Lagrangian
\[
L : R^{p+n+p} \to R, \left(t^\alpha, x^i, x^i_\alpha\right) \to L \left(t^\alpha, x^i, x^i_\alpha\right)
\]
determines the Euler-Lagrange equations
\[
\frac{\partial}{\partial t^\alpha} \frac{\partial L}{\partial x^i_\alpha} = \frac{\partial L}{\partial x^i}, \ i = 1, ..., n, \ \alpha = 1, ..., p
\]
(PDEs system of second order in the n-dimensional space). We remark that in the left hand member we have summation after the index \( \alpha \) (trace).

4.2 An action that produces Poisson-gradient systems

Let \( \alpha = 1, ..., p, i = 1, ..., n, u^i : T_0 \to R, t = (t^1, ..., t^p) \to u^i (t^1, ..., t^p), u : T_0 \to R^n, u(t) = (u^1(t), ..., u^n(t)), u^i_\alpha = \frac{\partial u^i}{\partial t^\alpha}, u^i_\alpha = (u^i_\alpha) \).

We consider the Lagrangian
\[
L : T_0 \times R^n \times R^{np} \to R, \left(t^\alpha, u^i, u^i_\alpha\right) \to L \left(t^\alpha, u^i, u^i_\alpha\right),
\]
\[
L \left(t^\alpha, u^i, u^i_\alpha\right) = \frac{1}{2} \left|\frac{\partial u^i}{\partial t}\right|^2 + F(t, u(t)).
\]

A function \( u \) (field) that realizes the minimum of the action
\[
\varphi (u) = \int_{T_0} L \left(t, u(t), \frac{\partial u}{\partial t}(t)\right) dt^1 \wedge ... \wedge dt^p,
\]
verifies a PDEs system of Poisson-gradient type (Euler-Lagrange equations on \( H^1_T \))
\[
\Delta u(t) = \nabla F(t, u(t)),
\]
together with the boundary conditions
\[
u \mid_{S^i_T} = u \mid_{S^i_T} \frac{\partial u}{\partial t} \mid_{S^i_T} = \frac{\partial u}{\partial t} \mid_{S^i_T}, i = 1, ..., p.
\]
4.3 Periodical solutions of Poisson-gradient dynamical systems with bounded non-linearity

**Theorem 4.** Suppose the function $F : T_0 \times R^n \to R$, $(t, u) \mapsto F(t, u)$ satisfies four properties:

1. $F(t, u)$ is measurable in $t$ for any $u \in R^n$ and it is continuously differentiable in $u$ for any $t \in T_0$.
2. There exist the functions $a \in C^1 (R^+, R^+)$ and $b \in C (T_0, R^+)$ such that for any $t \in T_0$ and any $u \in R^n$ to have $|F(t, u)| \leq a(|u|) b(t)$ and $|\nabla_u F(t, u)| \leq a(|u|) b(t)$.
3. It exists $g \in C^1 (T_0, R)$ such that for any $t \in T_0$ and any $u \in R^n$, to have $|\nabla_u F(t, u)| \leq g(t)$.

4. The action $\varphi_1 (u) = \int_{T_0} F(t, u (t)) dt^1 \land \ldots \land dt^p$ is weakly lower semi-continuous.

If $\int_{T_0} F(t, u) dt^1 \land \ldots \land dt^p \to \infty$ when $|u| \to \infty$, then the Dirichlet problem

$$\Delta u (t) = \nabla F(t, u (t)), \quad u |_{S_i^-} = u |_{S_i^+},$$

has at least a solution which minimizes the action

$$\varphi (u) = \int_{T_0} \left[ \frac{1}{2} \frac{\partial u}{\partial t} \right]^2 + F(t, u (t)) \right] dt^1 \land \ldots \land dt^p$$

in $H^1_T$.

**Proof.** We consider $u = \pi + \tilde{u}$, where $\pi = \frac{1}{T_{1} \ldots T_{p}} \int_{T_0} u (t) dt^1 \land \ldots \land dt^p$. Then

$$\varphi (u) = \int_{T_0} \left[ \frac{1}{2} \frac{\partial u}{\partial t} \right]^2 + F(t, u (t)) \right] dt^1 \land \ldots \land dt^p$$

$$= \int_{T_0} \left[ \frac{1}{2} \frac{\partial \pi}{\partial t} \right]^2 + F(t, \pi) - F(t, \pi) + F(t, u (t)) \right] dt^1 \land \ldots \land dt^p$$

$$= \int_{T_0} \left[ \frac{1}{2} \frac{\partial u}{\partial t} \right]^2 + F(t, \pi (t)) \right] dt^1 \land \ldots \land dt^p$$

$$+ \int_{T_0} (\nabla_u F(t, \pi + s \tilde{u} (t)), \tilde{u}(t)) ds \land dt^1 \land \ldots \land dt^p.$$
Also
\[ \| \tilde{u} \| = \left( \int_{T_0} \left( |\tilde{u}(t)|^2 + \left| \frac{\partial \tilde{u}}{\partial t}(t) \right| \right) dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}} \]
\[ = \left( \int_{T_0} \left( |\tilde{u}(t)|^2 + \left| \frac{\partial u}{\partial t}(t) \right| \right) dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}}. \]
With the Wirtinger inequality we obtain
\[ \| \tilde{u} \| \leq \left( \int_{T_0} \left( C \left| \frac{\partial \tilde{u}}{\partial t}(t) \right|^2 + \left| \frac{\partial u}{\partial t}(t) \right| \right) dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}} \]
\[ = (C + 1) \left( \int_{T_0} \left| \frac{\partial u}{\partial t}(t) \right|^2 dt^1 \wedge \ldots \wedge dt^p \right)^{\frac{1}{2}}. \]
The condition \( \| \tilde{u} \| \rightarrow \infty \) implies
\[ \int_{T_0} \left| \frac{\partial u}{\partial t}(t) \right|^2 dt^1 \wedge \ldots \wedge dt^p \rightarrow \infty. \]  
(6)

From the hypothesis and (5) or (6) it follows that if \( \|u\| \rightarrow \infty \), then \( \varphi(u) \rightarrow \infty \). So \( \varphi \) is a coercive application. This means that \( \varphi \) has a minimizing bounded sequence \( (u_k) \). The Hilbert space \( H^1 \) is reflexive. By consequence, the sequence \( (u_k) \) (or one of his subsequence) is weakly convergent in \( H^1 \) with the limit \( u \). Because
\[ \varphi_2(u) = \int_{T_0} \delta_{ij} \delta^{\alpha\beta} \frac{\partial u^i}{\partial \alpha}(t) \frac{\partial u^j}{\partial \beta}(t) dt^1 \wedge \ldots \wedge dt^p \]
is convex, it follows that \( \varphi_2 \) is weakly lower semi-continuous, so that the action
\[ \varphi(u) = \varphi_1(u) + \varphi_2(u) \]
is weakly lower semi-continuous and \( \varphi(u) \leq \lim_\{\varphi(u_k)\} \). This means that \( u \) is minimum point of \( \varphi \).

We build the function \( \Phi : [-1, 1] \rightarrow R \)
\[ \Phi(\lambda) = \varphi(u + \lambda v) \]
\[ = \int_{T_0} \frac{1}{2} \left| \frac{\partial}{\partial t}(u(t) + \lambda v(t)) \right|^2 \]
\[ + F(t, u(t) + \lambda v(t)) dt^1 \wedge \ldots \wedge dt^p, \]
where \( v \in C^{p\infty}_0 \). The point \( \lambda = 0 \) is a critical point of \( \Phi \) if and only if the point \( u \) is a critical point of \( \varphi \). Consequently
\[ 0 = \langle \varphi'(u), v \rangle = \int_{T_0} [\delta^{\alpha\beta} \delta_{ij} \frac{\partial u^i}{\partial \alpha} \frac{\partial v^j}{\partial \beta}] dt^1 \wedge \ldots \wedge dt^p, \]
for all \( v \in H^1 \) and hence for all \( v \in C^{p\infty}_0 \). According to the definition of the weak divergence, i.e.,
\[ \int_{T_0} \delta^{\alpha\beta} \delta_{ij} \frac{\partial^2 u^i}{\partial \alpha \partial \beta} \frac{\partial v^j}{\partial \beta} dt^1 \wedge \ldots \wedge dt^p = \]
\[ = - \int_{T_0} \delta^{\alpha\beta} \delta_{ij} \frac{\partial^2 u^i}{\partial \alpha \partial \beta} \frac{\partial v^j}{\partial \beta} dt^1 \wedge \ldots \wedge dt^p, \]
the Jacobi matrix function \( \frac{\partial u}{\partial t} \) has weak divergence (the function \( u \) has a weak Laplacian) and
\[ \triangle u(t) = \nabla^\dagger F(t, u(t)) \]
a.e. on \( T_0 \). Also, the existence of weak derivatives \( \frac{\partial u}{\partial t} \) and \( \triangle u \) implies that
\[ u|_{S^-} = u|_{S^+}, \frac{\partial u}{\partial t}|_{S^-} = \frac{\partial u}{\partial t}|_{S^+}. \]

**Remark.** If the function \( u \) is at least of class \( C^2 \), then the definition of the weak divergence of the Jacobian matrix \( \frac{\partial u}{\partial t} \) (or of the weak Laplacian \( \triangle u \)) coincides with the classical definition. This fact is obvious if we have in mind the formula of integration by parts
\[ \int_{T_0} \delta^{\alpha\beta} \delta_{ij} \frac{\partial u^i}{\partial \alpha} \frac{\partial v^j}{\partial \beta} dt^1 \wedge \ldots \wedge dt^p = \]
\[ = \int_{T_0} \delta^{\alpha\beta} \delta_{ij} \frac{\partial}{\partial \beta} \left( \frac{\partial u^i}{\partial \alpha} v^j \right) dt^1 \wedge \ldots \wedge dt^p - \]
\[ - \int_{T_0} \delta^{\alpha\beta} \delta_{ij} \frac{\partial^2 u^i}{\partial \alpha \partial \beta} \frac{\partial v^j}{\partial \beta} dt^1 \wedge \ldots \wedge dt^p. \]

**References**


