Some Mathematical Problems of the Geoid Determination in the Coastal Areas of the Gulf of Mexico

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Abstract: - This paper deals with the outline of some mathematical problems of the Geoid determination. It is the report of the ongoing long term research, which includes some results from statistical models and Stokes-Helmert approach as well as description of the expected future results.

Key-Words: - Geoid, spatial statistics, gravity, GPS, free-air anomalies, coastal areas

1 Introduction
The importance of knowing the exact Geoid [1, 2] is necessary for the success of different human activities as well as for our own survival. To know the precise Geoid may mean the difference between life and death in the case of tropical storms, floods, tsunamis, and other natural catastrophes. The other useful function of the Geoid is its positive impact on land issues like dredging, construction, and education, among others. By using satellites scientists have discovered the long wave (large area, small scale) Geoid for the Earth [3,4], but its precision is not sufficient when it comes to the relatively small areas at a large scale and/or local situations such as those created by hurricanes like Katrina, Rita (2005), and Ike (2008) in the coastal areas of the Gulf of Mexico. So, it is quite timely to develop more efficient methods and models of Geoid determination at the local level based on local observations of gravity and complimented by observations of gravity from the air and space. In today’s satellite age the height can be determined within just a few centimeters of accuracy geometrically by the global positioning system (GPS). If geoid models reach the same accuracy, local, national, or global vertical systems can be established in a quick and economical way and produce a Geoid with 1cm accuracy almost anywhere. This is the dream of the geodesists and geoscientists.

2 Problems
It is well known that the greatest errors in the geoid determination [5] happen either in mountainous terrain - due to sharp variations in the relief - or in coastal areas when two large bodies of water and land of quite different nature and density are brought together. This statement brings us to the main goal of this research. The main objective of this ongoing research is to devise mathematical models and computational methods to achieve the best possible precision - pending the quality and quantity of data – for the determination of the Geoid in coastal areas of the Gulf of Mexico. We would like to outline five mathematical problems which are strongly connected with the main objective of our research to determine a
sufficiently precise Geoid (especially short wave, local Geoid) and to evaluate errors in the Geoid determination. The solution of even some of these five problems in itself would be a great intellectual contribution to the field of applied mathematics and physical geodesy as determined by the International Association of Geodesy. In addition, which is more important, we will find ways to minimize all these errors with the goal of achieving a 1cm precise Geoid (at least on the local level, if possible) for coastal areas. Our team of one mathematician, one geoscientist, and two computer scientists is committed to solve these five problems to achieve the main goals.

1. Evaluation of precision when the Geoid approximation is obtained by using free air anomalies (FAA) and kriging models. This work is ongoing and some results are presented in the following section.

2. Solution of Laplace equation: evaluation of errors with boundary conditions. We will estimate the number of terms in the series solution of the Laplas Equation of the gravitational potential in the elliptic coordinates to get the needed precision of the Geoid, with its given boundary conditions. Using such an approach, along with using an e-Cluster computing system, will result in improvements to existing parallel processing algorithms for the solution of the boundary value problem utilizing huge gravity data bases that are currently available.

3. Evaluation of the maximum possible errors and evaluation of the confidence intervals of such errors in coastal areas where there is a boundary between two large masses with rather different densities.

4. Mathematical models based on Stokes-Helmert integral will be used to evaluate equipotential surfaces in coastal areas, find plumb lines to such surfaces, and determine errors in going upward and downward along these plumb lines. Knowing the behavior of equipotential surfaces in coastal areas can assist in determining plumb lines to such surfaces at different levels and, consequently, estimate their length from any given altitude.

5. Evaluation of errors in the solution of the Laplace equation, when both, boundary surface and boundary values have some initial errors. This is quite a complicated mathematical problem; we want to estimate the maximum errors of the solution of the Laplace Partial Derivative Equation when boundary values and the boundary itself are not known precisely.

There are also sub-objectives that have their own intrinsic value, these sub-objectives form the necessary steps and components of the main objectives of this research: i) Mathematical proof that there are equations which, if applied to perfectly observed gravity data for Earth, yield the location of the geoid accurately to within 1 cm. ii) Determination of the quality and quantity of data necessary in order to properly use these equations, if accuracy is given. iii) Non-linear effects of the geodetic boundary value problems on Geoid determinations. iv) Exploration of modern numerical methods in solving the geodetic boundary value problem of the gravity field (domain decomposition, finite elements, and others). v) Studies on data requirements, quality, distribution and density, for accurate
Determination of the Geoid in some coastal areas of the Gulf of Mexico by using actual gravity data to find those areas that are the most susceptible to floods and storm surge created by tropical storms and hurricanes.

In today’s satellite age, height can be determined within just a few centimeters of accuracy geometrically by a global positioning system (GPS) relative to the mathematical ellipsoid surface. If geoid models reach the same accuracy, local, national, or global vertical systems can be established in a quick and economical way and produce elevations relative to the geoid (sea level) with desired accuracy almost anywhere. Geoid modeling has been based on Stokes [6] and Molodensky’s [7] theories. In both theories, including the theories of gravity and topography reductions, which are of fundamental importance for determination of the precise geoid, a lot of assumptions have to be made to achieve the desired results. Due to the massive and still improving knowledge of the Earth’s surface, fixed-boundary value problems seem more adequate than the theoretical approach. Numerical studies along this line are not only important for practice, but also may cause a fundamental change in physical geodesy.

### 3 Kriging Results

One of the approaches mentioned above – already in practice – is the kriging method of spatial statistics. We have used data for the free air anomalies (FAA) of gravity supplied by the Naval Research Lab’s flights along meridians and parallels (see data sample presented in Figure 1.) to represent a continuous map of gravity anomalies by using kriging methods (see Figure 2.). Then by the application of the Stokes-Helmert Integral we can estimate the downward surface of the Geoid with a precision of 2-3 inches. In the preliminary results on the ongoing research in the mathematical and statistical modeling to determine the precise Geoid and estimate errors and their confidence intervals, we presented the estimation of errors derived from the application of the spatial statistics, namely kriging, to the free airborne anomalies (FAA) provided by the Naval Research Lab. The process described here focuses on two types of claims about errors of in Geoid determination. Firstly, claims about the maximum error in the Geoid undulation with given level of confidence. Secondly, claims of the least greatest error and reliability of such an error. Initial investigations show promising results on the Geoid precision by using a reasonably dense set of FAA data. So, here we have two kinds of problems from the point of view of mathematical and statistical modeling. Firstly, we want to minimize the greatest error $E_\alpha$ for any given confidence $1-\alpha$. Secondly, we would like to minimize such a confidence interval whenever possible.

The results of our preliminary statistical analysis can be seen in Figure 1 [8]. Furthermore, the set of FAA data obtained during flights along parallels with an interval of 30’ and along meridians with an interval of 12’. At the beginning of our investigation, we used data consisting of gravitational anomalies observed in the Gulf of Mexico. The data provided an opportunity to disregard the complexities of the downward extensions that resulted from complex terrains.
The computations were accomplished using Matlab Spatial Statistics toolbox EasyKrig v. 3.0. The next two figures show the linear variogram and the variogram map which will be used to determine errors with the given confidence level. Figure 2 illustrates the variogram for FAA data. As illustrated in the figure above, the approximation of the distance variance by linear function is quite successful, and, in our case, it is much better than a Gaussian or an exponential approximation. The variance map is presented in Figure 3. Here one can observe that the variation of free air anomalies, in the middle of the region, is slightly greater. This is due to the loss of data at some northern ends of the meridian tracks of the flights. This figure allows us to evaluate the confidence interval for an error of the gravitational anomaly at any point in the region under consideration. For instance, along the equal variance lines of 0.0548 (the greatest variance in the central part of the variance map on figure 3) and $\alpha=0.05$ we can claim a confidence level of 95% that errors for gravitational anomaly are within the interval (-0.107;+0.107). In other parts of the region, we can claim smaller errors for the same confidence within the intervals for errors within the interval (-0.057;+0.057). These numbers could give us the rough estimation for a possible error in the Geoid determination. For the International 1924 ellipsoid, the second approximation of the height correction is well know (Heiskanen and Moritz, 1967, 80). In the first approximation, the height correction is $0.3086 h$ mGal. This is the standard and routinely used approximation for height correction. It is commonly called the “free-air” correction. It is possible to find a very rough estimate for the maximum possible error in the precision of the Geoid if we use the kriging method to find the free-air gravitational anomalies. Then by applying the discrete Stokes-Harte method we can determine the Geoid. Using this approach we can determine that the errors are somewhere under 15 cm. Figure 4 illustrates the resulting map for FAA when applying the kriging approach to the initial data which has been used in the Stokes-Helmert method.
This approach allows us the opportunity to evaluate the Geoid undulations from the ellipsoid of reference and give the evaluation of the errors in the worst possible scenarios.

4 Some Equipotential Surfaces
Another aspect of the project in progress deals with the modeling of the equipotential surfaces for the ideal in coastal areas. We have already found such surfaces around the land-water interface by using the Stokes-Helmert integral [1,2,7]. The next three figures represent equipotential levels and gravitational potentials in 2D and 3D view: Knowledge of the behavior of the equipotential surfaces in coastal areas provides us an opportunity to estimate the greatest possible errors by using the behavior of the rates of change, or, in other words, partial derivatives, of such surfaces at different altitudes (levels).

Additionally, equipotential surfaces provide us with needed information about plumb lines, which in turn would be very helpful in the downward continuation as well as in better use of the famous 0.3068 correction factor to determine relative elevations.

5 Future Plans
These studies have been undertaken in conjunction with the Geoid study group under Commission 2 of International Association of Geodesy and U.S. National Geodetic Survey. Currently, the research discussed here focuses on aspects of the Geoid computation including geodetic relevant theories and applications. Our goal is to provide a framework of mathematical models consisting of theories and computational methods that ensure that a 1-2 cm accurate geoid can be achieved. We are especially interested in the precise determination of the Geoid in the coastal areas of the Gulf of Mexico which will help us determine which areas are the most susceptible to floods in the case of tropical storms and hurricanes storm surge events.

During this project, we expect to finish the evaluation of the maximum error produced by the kriging method approach mentioned in the previous section. We will estimate the confidence
intervals for small errors, and then we will try to solve optimization problems of minimizing the confidence interval for the given precision; and/or problem of minimizing precision for the given level of confidence [9].

The next step will be to start solving the boundary value problem for the gravity field assuming that we know the boundary with given precision as well as gravity on this given boundary using some other precision. The goal is to evaluate the gravity field by solving the Laplace boundary value problem. Solving this equation in the forms of Legendre polynomials of the 1st and the 2nd kind is very helpful in determining the exact Geoid and estimation of possible errors [10]. We have also started preliminary work on visualization. Visualization can be a powerful tool for both disaster recovery and disaster planning. Using an accurate local Geoid model will make the visualization more reliable, thus increasing the quality of the planning, and increasing the success of recovery. In addition, a more reliable model will allow for better predictions of how much damage will be caused by such events. A second area where visualization will be used in this project is in improving the model itself. The mathematical model will only be an approximation of the Geoid. The resolution of the model will depend on many factors, including the accuracy and resolution of gravity measurements. A visualization that shows the difference between the predicted Geoid with actual measurements can be a powerful tool in showing where the model has acceptable error, and where it does not.

References: