A Class of Alternating Group Finite Difference Method For solving Convection-Diffusion Equation

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Abstract: In this paper, with an exponential-type transformation, a class of alternating group finite difference method(AGE-I) based on the Saul’yev asymmetric schemes and the classical explicit-implicit schemes is derived for solving convection-diffusion equation. The method is suitable for parallel computation, unconditionally stable, and is of high accuracy especially when ε is small. Furthermore, a class of alternating group method(AGC-N) is derived by placing the Crank-Nicolson scheme at the fit situation. In order to verify the present method, numerical experiments are given at the end of the paper.

Key-Words: convection-diffusion equation, finite difference, parallel computation, exponential-type transformation, alternating group

1 Preface

We will consider the following convection-diffusion equation:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + k \frac{\partial u}{\partial x} &= \varepsilon \frac{\partial^2 u}{\partial x^2}, & 0 \leq x \leq 1, & 0 \leq t \leq T \\
u(x,0) &= f(x), & u(0,t) &= g_1(t), & u(1,t) &= g_2(t).
\end{aligned}
\]

Many finite difference methods have been presented for solving (1.1) so far, which are classified into three categories: The explicit, implicit and semi-implicit methods. Most of explicit methods are short in stability and accuracy, while implicit methods are unadaptable for parallel computing, and need to solve large system of equations. D. J. Evans presented an alternating group explicit method (AGE) by the specific combination of several asymmetric schemes in [1,2]. Because of the parallelism and absolute stability, the AGE method is widely cared and developed by many authors such as Baolin Zhang, Zhiyue Zhang etc in [3-7], while Rohallah Tavakoli derived a class of domain-split method based on the AGE method for 1D and 2D diffusion equation in [8]. Most of the developed methods have the same advantage of good stability and parallelism, but have difficulty of computation in the case of small ε. Zhenfu Tian presented a new group explicit method using an exponential-type transformation in [9], which has advantage of solving the convection-diffusion equation with small ε, and has higher accuracy than the AGE method presented by D. J. Evans.

The construction of this paper is as follows: In section 2 of this paper, an exponential-type transformation is used to get the integral conservative form of convection-diffusion equation, and a class of new unconditionally alternating group explicit conservative finite difference method (AGE-I) is derived using the Saul’yev asymmetric schemes and the classical explicit-implicit schemes. In section 3, an alternating group Crank-Nicolson method (AGC-N) is derived. In section 4, stability analysis is given. In section 5, numerical experiments on stability and accuracy are presented, which show that two methods above are superior to the methods in [2,7,9].

2 The Alternating Group Explicit-Implicit (AGE-I) Method

The domain Ω : (0, 1) × (0, T) will be divided into \((m \times N)\) meshes with spatial step size \(h = \frac{1}{m}\) in \(x\) direction and the time step size \(\tau = \frac{T}{N}\). Grid points are denoted by \((x_i, t_n), x_i = ih (i = 0, 1, \cdots, m), t_n = n\tau (n = 0, 1, \cdots, N)\). The numerical solution of (1.1) is denoted by \(u^n_i\), while the exact solution \(u(x_i, t_n)\).

Following [9], the equation (2.1) is equivalent to

\[
e^{-\frac{k\varepsilon}{\tau} \frac{\partial}{\partial t}} u^n_i = \varepsilon \frac{\partial}{\partial x} \left( e^{-\frac{k\varepsilon}{\tau} \frac{\partial}{\partial x}} u^n_i \right).
\]

Integral from \(x_{i-\frac{1}{2}}\) to \(x_{i+\frac{1}{2}}\), then we have

\[
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} e^{-\frac{k\varepsilon}{\tau} \frac{\partial}{\partial t}} u^n_i dx = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \varepsilon \frac{\partial}{\partial x} \left( e^{-\frac{k\varepsilon}{\tau} \frac{\partial}{\partial x}} u^n_i \right) dx.
\]
We will also use the following classical explicit-difference schemes:

\[
\begin{align*}
\left( e^{\frac{kh}{\tau}} - e^{\frac{kh}{\tau}} \right) u_{i+1}^{n+1} - \frac{u_{i}^{n+1} + u_{i}^{n}}{2} = & \\
\left( e^{\frac{kh}{\tau}} - e^{\frac{kh}{\tau}} \right) u_{i}^{n+1} - \frac{u_{i}^{n+1} + u_{i}^{n}}{2} = & \\
\left( e^{\frac{kh}{\tau}} - e^{\frac{kh}{\tau}} \right) u_{i+1}^{n+1} - \frac{u_{i+1}^{n+1} + u_{i}^{n}}{2} = & \\
\left( e^{\frac{kh}{\tau}} - e^{\frac{kh}{\tau}} \right) u_{i}^{n+1} - \frac{u_{i+1}^{n+1} + u_{i}^{n}}{2} = & \\
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\left( e^{\frac{kh}{\tau}} - e^{\frac{kh}{\tau}} \right) u_{i+1}^{n+1} - \frac{u_{i+1}^{n+1} + u_{i}^{n}}{2} = & \\
\left( e^{\frac{kh}{\tau}} - e^{\frac{kh}{\tau}} \right) u_{i}^{n+1} - \frac{u_{i+1}^{n+1} + u_{i}^{n}}{2} = & \\
\end{align*}
\]

Let \( m = 1 = (2s_{1} + 1)s_{2} \), here \( s_{1} \) and \( s_{2} \) are integers. The purpose of the paper is to get the solution of the \((n + 1)\)-th and the \((n + 2)\)-th time level with the solution of the \(n\)-th time level known. Using the schemes mentioned above, we will have four basic point groups: (1)"GL" group: \( 2s_{2} \) inner points are involved, and (2.5), (2.10), …, (2.10), (2.6), (2.7), (2.10), …, (2.10), (2.8) are used respectively.

(2)"HR" group: \( s_{2} \) inner points are involved, and (2.5), (2.10), …, (2.10), (2.6) are used respectively.

(3)"HL" group: \( s_{2} \) inner points are involved, and (2.7), (2.9), …, (2.9), (2.8) are used respectively.

(4)"GR" group: \( 2s_{2} \) inner points are involved, and (2.5), (2.9), …, (2.9), (2.6), (2.7), (2.9), …, (2.9), (2.8) are used respectively.

Based on the basic point groups above, the alternating group method will be presented as follows:

First at the \((n + 1)\)-th time level, we will have \((s_{1} + 1)\) point groups. "GL" are used in the first \( s_{1} \) point groups respectively, while "HR" are used in the last point group. Second at the \((n + 2)\)-th time level, we will still have \((s_{1} + 1)\) point groups. "HL" are used in the First point group, while "GR" are used in the right \( s_{1} \) point groups. Let \( U^{n} = (u_{1}^{n}, u_{2}^{n}, \cdots, u_{m-1}^{n})^{T} \), we can denote the alternating group method as follows:

\[
\begin{align*}
(I + rG_{1})U^{n+1} &= (I - rG_{2})U^{n} + F_{1}^{n} \\
(I + rG_{2})U^{n+2} &= (I - rG_{1})U^{n+1} + F_{2}^{n}
\end{align*}
\]

here \( F_{1}^{n} \) and \( F_{2}^{n} \) are known vectors relevant to the boundary.

We will also use the following classical explicit-implicit schemes:

\[
u_{i+1}^{n+1} = ru_{i+1}^{n+1} + [1 - r(p + q)]u_{i+1}^{n} + rqu_{i}^{n+1}
\]
3 The Alternating Group Crank-Nicolson(AGC-N) Method

First we will present the Crank-Nicolson scheme for solving (1.1):

\[
[1 + r(\frac{p}{2} + \frac{q}{2})]u_i^{n+1} - \frac{rp}{2}u_{i+1}^{n+1} - \frac{rq}{2}u_{i-1}^{n+1} = \frac{rq}{2}u_i^{n-1} + [1 - r(\frac{p}{2} + \frac{q}{2})]u_i^n + \frac{rp}{2}u_{i+1}^{n+1}
\]

(3.1)

If we replace (2.9) and (2.10) with (3.1) in section 2, then we can derive the alternating group Crank-Nicolson method as follows:

\[
\begin{bmatrix}
(I + r\mathcal{C}_1)U_1^{n+1} = (I - r\mathcal{C}_2)U_1^n + \mathcal{F}_1^n \\
(I + r\mathcal{C}_2)U_2^{n+2} = (I - r\mathcal{C}_1)U_2^{n+1} + \mathcal{F}_2^n
\end{bmatrix}
\]

(3.2)

here \(\mathcal{F}_1^n\) and \(\mathcal{F}_2^n\) are also known vectors relevant to the boundary.
4 Stability Analysis

Theorem 1 The AGE-I method (2.11) is unconditionally stable.

Proof: Let $n$ be an even number, then we have

$$U^n = GU^{n-2} + F^{n-2} = G^2 U^0 + \sum_{k=1}^{n/2} F^{n-2k} G^{k-1}$$

here

$$G = (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1}(I - rG_2).$$

$F^{n-k}$ is definite vector, $F^{n-k} = (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1}F^n + (I + rG_2)^{-1}F^{n-k}, (k = 2, 4, \cdots, n)$. Obviously $G_1$ and $G_2$ are both diagonally dominant matrices, which shows $G_1$ and $G_2$ and are both nonnegative definite real matrices. From [10] we have:

$$\|G_1^{-1}\|_2 \leq 1, \|G_2^{-1}\|_2 \leq 1,$$

$$\|(I - rG_1)(I + rG_1)^{-1}\|_2 \leq 1, \|(I - rG_1)(I + rG_1)^{-1}\|_2 \leq 1.$$

Let $G = (I + rG_2)(I - rG_1)(I + rG_1)^{-1}(I - rG_2)(I + rG_2)^{-1}$, then we have $\rho(G) = \rho(G) \leq \|G\|_2 \leq 1$. Therefore $\|G\| \leq M$, here $M$ is an positive number, which shows the method presented by (2.11) is unconditionally stable.

Similarly we have:

Theorem 2 The AGC-N method (3.2) is unconditionally stable.

5 Numerical Experiments

Example 1: We consider the following problem:

$$\frac{\partial u}{\partial t} + k \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T,$$

$$u(x, 0) = 0,$$

$$u(0, t) = 0, u(1, t) = 1.$$

The exact solution of the problem above is denoted in [2] as below:

$$u(x, t) \approx \frac{e^{kt}}{e^{kt} - 1} \left( \sum_{n=1}^{\infty} \frac{(-1)^n \pi}{(n \pi)^2} \sin(n \pi x) e^{-[(n \pi)^2 \varepsilon + \frac{4}{k^2}]} t \right).$$

Let $A.E. = |u_n - u(x, t_n)|$ and P.E.$= 100 \times \frac{|u_n - u(x, t_n)|}{u(x, t_n)}$ denote maximum absolute error and relevant error respectively, while let $s_2 = 3, m = 16$. We compare the numerical results of (2.11) and (3.2) with the results in [2, 7, 9] in Table 1.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>0.01, $t = 120\tau, \varepsilon = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.E.(AGE-I)</td>
<td>8.785 x 10^{-7}</td>
</tr>
<tr>
<td>A.E.(AGC-N)</td>
<td>6.335 x 10^{-7}</td>
</tr>
<tr>
<td>A.E.[9]</td>
<td>1.298 x 10^{-6}</td>
</tr>
<tr>
<td>A.E.[7]</td>
<td>1.161 x 10^{-6}</td>
</tr>
<tr>
<td>A.E.(Evans)</td>
<td>3.799 x 10^{-5}</td>
</tr>
<tr>
<td>P.E.(AGE-I)</td>
<td>3.681 x 10^{-4}</td>
</tr>
<tr>
<td>P.E.(AGC-N)</td>
<td>2.717 x 10^{-4}</td>
</tr>
<tr>
<td>P.E.[9]</td>
<td>5.352 x 10^{-3}</td>
</tr>
<tr>
<td>P.E.[7]</td>
<td>4.183 x 10^{-3}</td>
</tr>
<tr>
<td>P.E.(Evans)</td>
<td>1.739 x 10^{-2}</td>
</tr>
</tbody>
</table>

Table 2: Numerical results of comparison $k = 1$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>0.0001, $t = 100\tau, \varepsilon = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.E.(AGE-I)</td>
<td>6.850 x 10^{-7}</td>
</tr>
<tr>
<td>A.E.(AGC-N)</td>
<td>3.951 x 10^{-7}</td>
</tr>
<tr>
<td>A.E.[9]</td>
<td>1.151 x 10^{-6}</td>
</tr>
<tr>
<td>A.E.[7]</td>
<td>9.564 x 10^{-7}</td>
</tr>
<tr>
<td>A.E.(Evans)</td>
<td>7.741 x 10^{-6}</td>
</tr>
<tr>
<td>P.E.(AGE-I)</td>
<td>8.155 x 10^{-1}</td>
</tr>
<tr>
<td>P.E.(AGC-N)</td>
<td>5.836 x 10^{-1}</td>
</tr>
<tr>
<td>P.E.[9]</td>
<td>5.243</td>
</tr>
<tr>
<td>P.E.(Evans)</td>
<td>7.488</td>
</tr>
</tbody>
</table>

Example 2: Consider the equation

$$\frac{\partial u}{\partial t} + k \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad -L \leq x \leq L, t > 0$$
with the boundary condition(L=2):

\[ u(0, t) = 1.0, \quad u(2, t) = 0.0, \quad t > 0. \quad (5.2) \]

and the initial condition:

\[ u(x, 0) = \begin{cases} 
1.0, & -2 \leq x < 0, \\
0.5, & x = 0, \\
0.0, & 0 < x \leq 2.
\end{cases} \quad (5.3) \]

The exact solution of the problem above is denoted as below:

\[ u(x, t) = \frac{1}{2} \sum_{m=1}^{n} \sin[(2m-1) \pi (x - kt) / L] \exp(-\varepsilon (2m-1)^2 \pi^2 t / L^2) / (2m-1). \]

We compare the present method with the methods in [2,7,9].

Table 3: Numerical results of comparison at \( m = 154, \tau = 10^{-4}, k = 1, \varepsilon = 1 \)

<table>
<thead>
<tr>
<th>Method</th>
<th>( t = 1000\tau )</th>
<th>( t = 2000\tau )</th>
<th>( t = 3000\tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P.E.(AGE-I)</td>
<td>6.815 \times 10^{-4}</td>
<td>9.008 \times 10^{-4}</td>
<td>8.507 \times 10^{-3}</td>
</tr>
<tr>
<td>P.E.(AGC-N)</td>
<td>6.314 \times 10^{-4}</td>
<td>4.381 \times 10^{-4}</td>
<td>4.187 \times 10^{-3}</td>
</tr>
<tr>
<td>P.E.<a href="Evans">2</a></td>
<td>4.262 \times 10^{-2}</td>
<td>7.323 \times 10^{-2}</td>
<td>8.952 \times 10^{-1}</td>
</tr>
<tr>
<td>P.E.[9]</td>
<td>8.216 \times 10^{-3}</td>
<td>1.357 \times 10^{-1}</td>
<td>3.568 \times 10^{-1}</td>
</tr>
<tr>
<td>P.E.[7]</td>
<td>2.931 \times 10^{-3}</td>
<td>5.648 \times 10^{-2}</td>
<td>3.429 \times 10^{-2}</td>
</tr>
</tbody>
</table>

Example 3: Consider the equation

\[ \frac{\partial u}{\partial t} + k \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \quad -2 \leq x \leq 2, \quad t > 0 \]

with the boundary condition:

\[ u(-2, t) = u(2, t) = -\sin(\pi kt) e^{-\varepsilon \pi^2 t}, \quad t > 0, \quad (5.4) \]

and the initial condition:

\[ u(x, 0) = \sin(\pi x). \quad (5.5) \]

The exact solution of the problem above is denoted as below:

\[ u(x, t) = \sin[\pi (x - kt)] e^{-\varepsilon \pi^2 t} \]

We also compare the numerical results of the present method with the methods in [2,7,9].

Table 4: Numerical results of comparison at \( m = 154, \tau = 10^{-4}, k = 1, \varepsilon = 0.01 \)

<table>
<thead>
<tr>
<th>Method</th>
<th>( t = 1000\tau )</th>
<th>( t = 2000\tau )</th>
<th>( t = 3000\tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P.E.(AGE-I)</td>
<td>Invalid</td>
<td>Invalid</td>
<td>Invalid</td>
</tr>
<tr>
<td>P.E.(AGC-N)</td>
<td>Invalid</td>
<td>Invalid</td>
<td>Invalid</td>
</tr>
<tr>
<td>P.E.<a href="Evans">2</a></td>
<td>Invalid</td>
<td>Invalid</td>
<td>Invalid</td>
</tr>
<tr>
<td>P.E.[9]</td>
<td>Invalid</td>
<td>Invalid</td>
<td>Invalid</td>
</tr>
<tr>
<td>P.E.[7]</td>
<td>Invalid</td>
<td>Invalid</td>
<td>Invalid</td>
</tr>
</tbody>
</table>

From table 1,2,3,4 we can see that the AGE-I method and the AGC-N method are of higher accurate than the methods in [2,9,10] especially when \( \varepsilon \) is small. On the other hand, we notice the two alternating group methods are both suitable for parallel computing obviously, and are unconditionally stable.

References: