Weighted inequalities with Hölder norms of solutions to \( Lu = \text{div} f \) in non-smooth domains

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Abstract: The rate of change of a solution to \( Lu = \text{div} f \) in a bounded, non-smooth domain \( \Omega \) in \( \mathbb{R}^d, d \geq 3, u|_{\partial \Omega} = 0 \) is investigated using a local Hölder norm of \( u \) and different measures on \( \Omega \). Results for

\[
L = \sum_{i,j=1}^{d} \frac{\delta}{\xi_i} (a_{ij}(x) \frac{\delta}{\xi_j})
\]

are presented.

Keywords: elliptic equations, Lipschitz domains, Borel measures, semi-discrete Littlewood-Paley type inequalities

1 Introduction: Questions concerning the rate of change of a temperature function or of a potential function in a limited environment are of fundamental importance in many applications of mathematics. For solutions to the heat equation and for harmonic functions in the upper half space, Wheeden and Wilson [WW] proved necessary and sufficient conditions on two Borel measures, one, \( \mu \), defined in \( \mathbb{R}^{d+1} \), the other, \( v(x')dx' \), defined on \( \mathbb{R}^d \), so that

\[
\left( \int_{\mathbb{R}^{d+1}} |\nabla u(x)|^q d\mu(x) \right)^{1/q} \leq C \left( \int_{\mathbb{R}^d} |f(x')|^p v(x') dx' \right)^{1/p}
\]

for any solution to the Dirichlet problem

\[
Lu = 0 \text{ in } \mathbb{R}^{d+1}, u = f \text{ on } \mathbb{R}^d.
\]

Here \( L = \Delta \) or \( \partial/\partial t - \Delta \), and \( 1 \leq p \leq q < \infty \) with \( q \geq 2 \). Later work by Sweezy and Wilson established analogous results for solutions to more general second order equations on bounded Lipschitz and on Lip(1,1/2) domains, [S1, SW1, SW2, W]. In this paper questions concerning the rate of change of solutions to the inhomogeneous equation \( Lu = \text{div} f \) in \( \Omega \), \( u|_{\partial \Omega} = 0 \) will be investigated for operators of the form

\[
L = \sum_{i,j=1}^{d} \frac{\delta}{\xi_i} (a_{ij}(x) \frac{\delta}{\xi_j}), \text{ with } L
\]

being symmetric and strictly elliptic, and \( \Omega \) a bounded domain in \( \mathbb{R}^d \) whose boundary satisfies an exterior cone condition. In [S2] the question of finding conditions on two measures \( \mu \) and \( v \), defined on \( \Omega \), to give the inequality

\[
\left( \int_{\mathbb{R}^{d+1}} |\nabla u(x)|^q d\mu(x) \right)^{1/q} \leq C \left( \int_{\mathbb{R}^d} \left( |f(x')|^p + \left| \text{div} f(x') \right|^p \right) dv(x') \right)^{1/p},
\]

was considered. It was shown that a condition involving a singular potential of the measure \( \mu \) gives the same kind of norm
inequality with a local H{"o}lder norm of the solution $u(x)$ replacing $|\nabla u(x)|$. The first result that will be proved here will be to define conditions on $\mu$ and $\nu$, less restrictive than the conditions previously studied in connection with obtaining a norm inequality involving $\nabla u(x)$, that give the following result:

$$\left( \int_{\Omega} \left( \|u\|_{p(x)} \right)^{q} \, d\mu(x) \right)^{1/q} \leq C \left( \int_{\Omega} \left( \left| f(x) \right|^{p} + \left| \text{div} f(x) \right|^{p} \right) \, d\nu(x) \right)^{1/p},$$

where $\|u\|_{p(x)} = \sup_{y \in P_{x}(100)} \frac{|u(x) - u(y)|}{|x-y|^d}$, $x \neq y$. $P_{x}(100)$ is a small cube centered at $x$, whose side length is, say, less than or equal to $(1/(100))$ of the distance from $x$ to $\partial \Omega$. (distance(x, $\partial \Omega$) $\equiv \delta(x))$. The strategy is to obtain a condition on Whitney type cubes in $\Omega$ for $\mu$ and $\nu$ that does not involve a singular potential and that does not require a reverse Hölder condition. In fact, the condition in Theorem A given below is similar to another of the conditions introduced previously [S3]. One more advantage of dealing with the H{"o}lder norm instead of the gradient of the solution is that we obtain the norm inequality on the domain for a much bigger range of exponents, $p$ and $q$. To prove Theorem A we must use a square function result which is given in Theorem B. Theorems A and B were announced in a talk at the International Conference of Applied Mathematics and Computing at Plovdiv, Bulgaria, August, 2008.

2 Problem Formulation: To state Theorem A we need several definitions. First, $W$ will denote the collection of certain Whitney-type dyadic cubes (these are dyadic cubes whose dimension compares with the cube’s distance from the boundary of $\Omega$) that lie in $\Omega$. These cubes have the property that their interiors are pair-wise disjoint; a fixed dilate of any such cube will also be a Whitney-type cube with respect to $\Omega$, and $\Omega = \bigcup_{Q \in W} Q_{j}$ is a large dyadic cube that contains $\Omega$. The cubes in $W$ are dyadic sub-cubes of $Q_{0}$. The measures $\mu$ and $\nu$ will be taken to be Borel measures; $\mu$ is defined on $\Omega$, with $\nu$ defined on $Q_{0}$, and $\nu$ is absolutely continuous with Lebesgue measure.

Next we define $M(Q_{j})$ for any dyadic cube $Q_{j}$ that lies inside $\Omega$, and for

$$d\sigma(y) = \left( \frac{d}{d\nu(y)} \right)^{1/p'} d\nu,$$

$$M(Q_{j}) = \max_{Q_{j} \subseteq Q_{0}} \left\{ \left( \frac{1}{|Q_{j}|} \int_{Q_{j}} \left( \frac{d\nu}{dx}(x) \right)^{p/(p'-s)} \, dx \right)^{-(d-s)/2} \, d\sigma(y) \right\}.\right.$$
The proof of Theorem A follows the same general outline initiated in [WW]; one employs a dual operator argument that depends on a Littlewood-Paley type inequality. Two standard results on Holder continuity for solutions to \( Lu = \text{div} \, \mathbf{f} \) will be used in the proof of Theorem A. They are paraphrased here in Lemmas 1 and 2 (see Gilbarg and Trudinger [GT], Chapter 8).

**Lemma 1:** \( L \) is strictly elliptic with coefficients bounded and measurable, and \( \mathbf{f} \in L^{s}(\Omega) \) for some \( s > d \). If \( Lu = \text{div} \, \mathbf{f} \) in \( \Omega \) with \( u \in W^{1,2}(\Omega) \), then for any \( \gamma > 1 \), \( B_{\gamma R}(y) \subset \Omega \), and any \( m > 1 \),

\[
\sup_{x \in B_{R}(y)} |u(x)| \leq C \left( \int_{B_{3R}(y)} |u(x)|^{m} \, dx \right)^{1/m} \]

\[
+ C \left( \frac{1}{\gamma} \right)^{d-s} \left\| \mathbf{f} \right\|_{L^{s}(\Omega)} \). \]

With the same hypotheses for \( L, \mathbf{f}, \) and \( u \) as in Lemma 1,

**Lemma 2:** For any \( B_{R_{0}}(y) \subset \Omega \), and any \( \gamma \),

\[\text{osc } u = \sup_{x \in \overline{B_{\gamma}(y)}} |u(x) - u(z)| \leq C \left( \frac{1}{\gamma} \right)^{s} \sup_{x \in \overline{B_{R_{0}}(y)}} |u(x)| \]

\[+ CR^{d} \left( \frac{1}{\gamma} \right)^{1/(1-s)} \right\| \mathbf{f} \right\|_{L^{s}(\Omega)} \]. \]

Note: \( \alpha = \alpha(d,s) \). The constant \( C \) in Lemma 2 depends on \( R_{0} \); however, this can be changed to a dependence on the diameter of \( \Omega \). Taking \( \Omega \) variously as itself, \( B_{\gamma R_{0}}(y) \) in \( \Omega \) with \( \gamma > 1 \), or as a fixed dilate of \( Q_{j} \), say \( \Omega = 4Q_{j} \) (which has the property that it is still a Whitney-type cube in the original domain), and taking \( m = 2 \), gives the form of these results used for local estimates below.

### 3 Problem Solution

**Outline of the proof of Theorem A:** As usual we start by writing \( \int_{\Omega} \| u \|_{L^{p}(\Omega)}^{p} \, d\mu(x) \) as the sum of integrals over the Whitney-type cubes in \( W \):

\[
\int_{\Omega} \| u \|_{L^{p}(\Omega)}^{p} \, d\mu(x) = \sum_{Q_{j} \in W} \int_{Q_{j}} \| u \|_{L^{p}(\Omega)}^{p} \, d\mu(x) .
\]

This expression can be shown to be dominated by

\[
\frac{C \mu(Q_{j})}{\mu(Q_{j})^{d-p}} \left( \frac{1}{\mu(Q_{j})} \int_{(3/2)Q_{j}} |u(x) - \overline{u}(x)|^{2} \, dx \right)^{1/2} \]

\[+ \left( \frac{1}{\mu(Q_{j})} \int_{(3/2)Q_{j}} |\overline{u}(x)|^{2} \, dx \right)^{1/2} \gamma \]

\[+ C \mu(Q_{j}) \left( \int_{Q_{j}} |\mathbf{f}(x)| \, dx \right)^{1/2} \]

\[+ \left( \int_{3Q_{j}} |\mathbf{f}(x)| \, dx \right)^{1/2} \gamma .
\]

Here \( u^\gamma(x) \) is the solution to \( Lv = \text{div} \, \mathbf{f} \) in \( 4Q_{j} \) \( , \) \( v(x') = 0 \) on \( \partial(4Q_{j}) \). A series of elementary estimates shows that we need only bound

\[
\sup_{\| \mathbf{f} \|_{L^{s}(\Omega)}} \mu(Q_{j}) \| u(Q_{j}) \|_{L^{s}(\Omega)} \cdot
\]

\[\left( \frac{1}{\mu(Q_{j})} \int_{(3/2)Q_{j}} |u(x) - \overline{u}(x)|^{2} \, dx \right)^{1/2} \] and

\[
\sup_{\| \mathbf{f} \|_{L^{s}(\Omega)}} \mu(Q_{j}) \| u(Q_{j}) \|_{L^{s}(\Omega)} \cdot
\]

\[\left( \frac{1}{\mu(Q_{j})} \int_{4Q_{j}} |\mathbf{f}(x)| \, dx \right)^{1/2} \] by a constant multiplying

\[\left( \int_{\Omega} \left( |\mathbf{f}(x)|^{p} + \left| \text{div} \, \mathbf{f}(x) \right|^{p} \right) \, d\nu(x) \right)^{1/p}, \]

and we will be done. The second sum can be shown to be less than or equal to
To show that \( \sum_{Q_j \in \mathcal{W}} \left( \frac{\mu(Q_j)g(Q_j)}{l(Q_j)^{q'}} \right) \) is dominated by \( \left( \sum g(Q_j)^{q'} \mu(Q_j) \right)^{\frac{1}{q'}} \), we exploit the fact that the sequence \( \{g(Q_i)\} \) can be assumed to have only finitely many non-zero terms (see [WW]). Moreover, \( q' \geq p \), so \( p' \geq q' \) and \( ((q')/(p')) \leq 1 \). This means that

\[
\sum_{Q_j \in \mathcal{W}} \left( \frac{\mu(Q_j)g(Q_j)}{l(Q_j)^{q'}} \right) \leq \sum_{Q_j \in \mathcal{W}} \left( \frac{\mu(Q_j)g(Q_j)}{l(Q_j)^{q'}} \right) \leq \sum_{Q_j \in \mathcal{W}} \left( \frac{\mu(Q_j)g(Q_j)}{l(Q_j)^{q'}} \right) \leq \sum_{Q_j \in \mathcal{W}} \left( \frac{\mu(Q_j)g(Q_j)}{l(Q_j)^{q'}} \right).
\]

It will suffice to have the term by term comparisons

\[
\mu(Q_j)^{q'} g(Q_j)^{q'} l(Q_j)^{(1-\frac{4}{q'}-a)q'}.\]

\[
\left( \int_{4Q_j} \left( \frac{dx}{dx} \right)^{\frac{q'}{p'}} dx \right)^{\frac{1}{q'}} \leq C \mu(Q_j) g(Q_j)^{q'}.
\]

Rearranging terms and taking \( q' \) roots on both sides of the resulting inequality gives the sufficient condition

\[
\mu(Q_j)^{\frac{1}{2}} \left( \int_{4Q_j} \left( \frac{dx}{dx} \right)^{\frac{q'}{p'}} dx \right)^{\frac{1}{2}} \leq C l(Q_j)^{\frac{q'}{p'q'}}.
\]

To handle the first sum,

\[
\sup \|g(Q_j)\|^{\frac{1}{q'}} - 1 \sum_{Q_j \in \mathcal{W}} \frac{\mu(Q_j)g(Q_j)}{l(Q_j)^{q'}}.
\]

\[
\left( \frac{1}{|Q_j|} \int_{(3/2)Q_j} |u(x) - \tilde{u}(x)|^2 dx \right)^{\frac{1}{q'}}
\]

one can write, taking \( G \) and \( G' \) to be the Green functions of \( \Omega \) and \( 4Q_j \) and \( M(x) = \int_{\Omega} \text{div} \bar{f}(y) \left( G(x,y) - \bar{G}(x,y) \right) dy, \)

\[
\left( \frac{1}{|Q_j|} \int_{(3/2)Q_j} |u(x) - \tilde{u}(x)|^2 dx \right)^{\frac{1}{q'}} \leq \sum_{Q_j \in \mathcal{W}} \mu(Q_j)^{q'} g(Q_j)^{q'} \leq \sum_{Q_j \in \mathcal{W}} \left( \frac{\mu(Q_j)g(Q_j)}{l(Q_j)^{q'}} \right).
\]

The first expression derives from writing the solution in the form

\[
u(x) = \int_{\Omega} \text{div} \bar{f}(y) G(x,y) dy,
\]

and likewise for \( u^*(x) \). The first inequality follows from Minkowski's inequality for integrals and the last inequality can be obtained by using Harnack's inequality on the non-negative solution \( G(x,y) - \bar{G}(x,y) \).
Next the order of summation and integration are interchanged to obtain

\[ \sum_{Q \in \mathcal{W}} \frac{\mu(Q) \mathbb{E}(Q)}{|Q|^\alpha} . \]

\[ \left( \frac{1}{|Q|} \int_{(3/2)Q} |u(x) - \mathbb{E}(x)|^2 \, dx \right)^{1/2} \leq \]

\[ \int_{\Omega} |\text{div} \tilde{f}(y)| \left| \sum_{Q \in \mathcal{W}} \frac{\mu(Q) \mathbb{E}(Q)}{|Q|^\alpha} \right| \cdot \]

\[ \left( \frac{1}{|Q|} \int_{(3/2)Q} \left| G(x,y) - \tilde{G}(x,y) \right| \, dx \right)^{1/2} \]

\[ = \int_{\Omega} |\text{div} \tilde{f}(y)| \left( \sum_{Q \in \mathcal{W}} \lambda_Q \varphi_Q(y) \right) \, dy \]

With \( \lambda_Q = \frac{\mu(Q) \mathbb{E}(Q)}{|Q|^\alpha} \) and

\[ \varphi_Q(y) = \frac{1}{|Q|} \int_{(3/2)Q} \left| G(x,y) - \tilde{G}(x,y) \right| \, dx. \]

At this point we will use the square function estimate provided by Theorem B.

**Theorem B:** Suppose that \( \tilde{f}(x) = \sum_{J \in \mathcal{F}} \lambda_J \varphi_J(x) \) is a function defined on \( \Omega \), where \( \mathcal{F} \) is a finite set of dyadic cubes from \( W \), and the \( \{ \varphi_J \}_{J \in \mathcal{F}} \) are a family of functions that satisfy conditions a), a'), b), and c), and such that \( \varphi_J(x) = 0 \) if \( x \in \Omega \setminus \mathcal{W} \). Then, if \( d\sigma \in A^p(Q_0, d\sigma) \), there is a constant \( C = C(d, \alpha, p, \Omega, \kappa, C_0) \) such that, for any \( 0 < p < \infty \),

\[ \|f\|_{L^p(Q_0, d\sigma)} \leq C \|g^*\|_{L^p(Q_0, d\sigma)}. \]

The proof of Theorem B and of a), a'), b), and c) is given in another paper [S3]. Next we need to define the function \( g^*(f)(x) = g^*(x) \) and to give the conditions a), a'), b), and c). When \( f(x) = \sum_{J \in \mathcal{F}} \lambda_J \varphi_J(x) \)

\[ g^*(x) = \left( \sum_{J \in \mathcal{F}} \frac{\lambda_J^2}{|Q_J|^\alpha} \left( 1 + \frac{|x-x_J|}{l(J)} \right)^{-\alpha} \right)^{1/2}. \]

\( g^*(x) \) is a discrete version of the \( g_\lambda^* \) function of classical Littlewood-Paley theory. The four conditions that will be assumed to hold for the family \( \{ \varphi_J(x) \} \) are (see the proof of Theorem B in [S3]):

\[ S(Q) = \{ J \in \mathcal{F} | J \subset Q \} \]

\[ a) \ |\varphi_J(x)| \leq C l(J)^{2-d/2} \left( 1 + \frac{|x-x_J|}{l(J)} \right)^{2-d}. \]

for all \( x \in \Omega \).

\[ a') \ |\varphi_J(x)| \leq C \delta(x) l(J)^{2-d/2} \left( 1 + \frac{|x-x_J|}{l(J)} \right)^{2-d}. \]

for all \( x \in \Omega \).

\[ b) \ |\varphi_J(x) - \varphi_J(y)| \leq C |x-y|^{\alpha} l(J)^{2-d/2}. \]

for all \( x \) and \( y \) in \( \eta Q \) and \( J \in S(Q) \).

\[ c) \ \left\| \sum_{J \in \mathcal{F}} \lambda_J \varphi_J(x) \right\|^2 \leq C \sum_{J \in \mathcal{F}} \lambda_J^2. \]

Assuming for now that the \( \varphi_J(y) \) satisfy the properties a), a'), b), and c), then Hölder’s inequality followed by the application of Theorem B to the function

\[ h(y) = \sum_{Q \in \mathcal{W}} \lambda_Q \varphi_Q(y), \]

gives

\[ \int_{\Omega} |\text{div} \tilde{f}(y)| \left( \sum_{Q \in \mathcal{W}} \lambda_Q \varphi_Q(y) \right) dy \leq \]

\[ C \|g^*(h)\|_{L^p(\Omega, d\sigma)}. \]

It will suffice to show that

\[ \|g^*(h)\|_{L^p(\Omega, d\sigma)} \leq C \|\langle g(Q) \rangle\|_{L^p(\Omega, d\sigma)}. \]

To utilize the fact that \( p > 2 \) implies \( p'/2 < 1, \)
we can assume that the sum defining \( h(y) \) is finite (see [WW]).

\[
\| g^*(h) \|_{L^p(Q_0, \partial \sigma)}^p = \int_{Q_0} \left( \sum_{i=1}^{\infty} \frac{x_i^j}{|Q_i|^{1/2}} \left( 1 + \frac{|y-x_i|}{l(Q_i)} \right)^{-\frac{d-\alpha}{2}} \right)^p d\sigma(y)
\]

\[
\leq \sum_{i=1}^{\infty} \frac{x_i^j}{|Q_i|^{1/2}} \cdot \int_{Q_0} \left( 1 + \frac{|y-x_i|}{l(Q_i)} \right)^{-\frac{(d-\alpha)p}{2}} d\sigma(y).
\]

So it suffices to show that the last sum is dominated by \( \left( \sum (g(Q_j)^q \mu(Q_j))^{p'/q} \right)^{p'/q} \).

Once again, taking advantage of the fact that \( q'/p' \leq 1 \), this is equivalent to showing that

\[
\sum_{i=1}^{\infty} \frac{x_i^j}{|Q_i|^{1/2}} \cdot \left( \int_{Q_0} \left( 1 + \frac{|y-x_i|}{l(Q_i)} \right)^{-\frac{(d-\alpha)p}{2}} d\sigma(y) \right)^{q/p'}
\]

\[
\leq \left( \sum (g(Q_j)^q \mu(Q_j)) \right)^{p'/q}.
\]

So if

\[
\frac{x_i^j}{|Q_i|^{1/2}} \left( \int_{Q_0} \left( 1 + \frac{|y-x_i|}{l(Q_i)} \right)^{-\frac{(d-\alpha)p}{2}} d\sigma(y) \right)^{q/p'}
\]

\[
\leq C(g(Q_j)^q \mu(Q_j)),
\]

we will have the desired result. But this is the same as requiring that

\[
\mu(Q_j)^{1/q} \left( \int_{Q_0} \left( 1 + \frac{|y-x_i|}{l(Q_i)} \right)^{-\frac{(d-\alpha)p}{2}} d\sigma(y) \right)^{1/p'}
\]

\[
\leq C l(Q_j)^{d-\alpha}.
\]

4 Conclusion Future work will entail finding conditions on two measures so that one can prove a weighted norm inequality and a semi-discrete Littlewood-Paley type inequality in the setting that is appropriate for the generalized heat equation; in other words, to find and prove analogues of both Theorem A and Theorem B for solutions to

\[
Lu = \text{div} \vec{f} \begin{cases} \text{in} \Omega, & |u|_{\partial \Omega} = 0, \end{cases}
\]

where \( L = \frac{\partial}{\partial t} - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \left( a_{ij}(x,t) \frac{\partial}{\partial x_i} \right) \), and \( \Omega \) is a domain with a rough boundary in \( \mathbb{R}^{d+1} \).

References:


