On Devaney’s Definition of Chaos for Discontinuous Dynamical Systems

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Abstract: One of the most widely accepted definition of chaos is the one by Devaney, which we will call Devaney-chaos. It consists of three conditions, (1) the sensitive dependence upon the initial condition, (2) the topological transitivity, and (3) the dense distribution of the periodic orbits. The third condition is often omitted for being too stringent. The purpose of our research is to investigate how the first two characteristic properties of Devaney-chaos are affected by the presence of the discontinuity, and subsequently, what kind of adjustments must be made to improve Devaney-chaos so that it can be applied to discontinuous dynamical systems as well as continuous systems. Under the afore-mentioned adjustments, we prove that the first two conditions of Devaney-chaos can be successfully used to characterize the complex orbit behavior of general 2-dimensional discontinuous dynamical systems. Also, we show that the straightforward application of unadjusted Devaney-chaos is too inclusive when the system is discontinuous, consequently necessitating the afore-mentioned adjustments. We use the classification theorems of the singularities of the piecewise isometric dynamical systems as the main tools.

Key–Words: Devaney-chaos, Discontinuous dynamics, Singularity.

1 Introduction

One of the most useful and widely accepted definition of chaos is the one by Devaney [4], which we will call Devaney-chaos (Definition 2.1). The concept of Devaney-chaos was developed for the iterative dynamics of continuous maps on continua. In this paper, we aim to study the effects of the discontinuity on the conditions of Devaney’s definition of chaos, and subsequently improve Devaney’s definition to the one that befits discontinuous dynamical systems as well as continuous systems. Figure 1.1 exemplifies the effects of the discontinuity we just mentioned. It illustrates the periodic sets (white) and the aperiodic sets (dark) of selected examples of the discontinuous dynamical systems.

There are a number of alternative ways to define the chaos, besides Devaney-chaos. An incomplete list of more popular methods include, Lyapunov-chaos (positive Lyapunov exponent) [1, 7, 21, 23, 25], topological chaos (positive topological entropy) [21, 23], Smale-chaos (homeomorphic to a Bernoulli shift dynamics)1 [1, 7], and Li-Yorke-chaos (the existence of

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1More common method is to look for the presence of Smale’s horse-shoe-map, but the horse-shoe-map is homeomorphic to a Bernoulli shift [1, 7].
non-trivial scrambled set) [5, 18, 19, 24]. Providing
the precise definitions of all these competing notions
of chaos and their comparative analyses is not our
concern here. The focus of our attention is Devaney-
chaos. We study and how it is affected, and therefore
must be adjusted, in the presence of the discontinuity.

Devaney-chaos is more inclusive than most of the
competing notions of chaos, especially when the dy-
namics includes singularity. Recent discoveries by
Goetz and Buzzi on the discontinuous dynamical sys-
tems include that the piecewise isometric dynamical
systems, which are partly inspired by the digital sig-
nal processing and Hamiltonian dynamics, can gen-
erate complicated orbit structure, even though their
Lyapunov-exponents and topological entropies are 0
[2, 8, 10]. Consequently, neither Lyapunov-chaos nor
topological chaos can be applied to explain the com-
plex behavior of the piecewise isometric dynamics.
Goetz also proved that Smale-chaos fails to apply as
well [9]. Devaney-chaos, on the other hand, proved to
be a useful tool, for some special cases, as exemplif-
yed by some of the author’s contributions in this topic
[12, 13, 14, 15, 16].

In general, however, the straightforward applica-
tion of Devaney-chaos can be too inclusive, as we will
see in Example 2.2. The main purpose of this paper
is to improve Devaney’s definition of chaos so that it
can be used for discontinuous dynamical systems as
well as for continuous systems (Definition 4.3). Also,
we show that our improved definition is well-defined
(Theorem 4.1) and as strong as possible (Theorem
3.3). We use the isometric continuation (Theorem 3.1)
as the main tool.

2 Devaney’s Definition of Chaos

In [4], whose first edition was published in 1989, De-
vaney defined chaos using three conditions, (1) the
sensitive dependence upon the initial condition, (2)
the topological transitivity, and (3) the dense distri-
bution of the periodic orbits. However, most of the
other literature admits only the first two conditions.
See, for instance, [1, 7, 21, 23, 25], for the alternative
descriptions of Devaney-chaos. We, too, will follow
the latter convention and concentrate our efforts only
to the first two conditions.

Definition 2.1. Let \((X, d)\) be a metric space. Then, a
map \(f : X \to X\) is said to be Devaney-chaotic on \(X\)
if it satisfies the following conditions.

\(f\) has sensitive dependence on initial conditions.
That is, there exists a certain \(\delta > 0\) such that, for any
\(x \in X\) and \(\epsilon > 0\), there exist some \(y \in X\) where
d\((x, y) < \epsilon\) and \(n \in \mathbb{N}\) so that
d\((f^n(x), f^n(y)) > \delta\).

(2) \(f\) is topologically transitive.
That is, for any pair of open sets \(U, V \subset X\), there
exists a certain \(n \in \mathbb{N}\) such that \(f^n(U) \cap V \neq \emptyset\).

Devaney’s definition of chaos makes sense only
for the iterative dynamical systems on continua. From
Definition 2.1, one can immediately see that no map
is Devaney-chaotic if \(X\) is a discrete space. By taking
\(\epsilon = \frac{1}{2}\), for instance, one can prove that no map in
\(\mathbb{Z}\) can have the sensitive dependence upon the initial
condition. For the rest of this paper, therefore, we will
assume that the metric space \(X\), on which the iterative
dynamics takes place, is a continuum.

The purpose of this article is to study Devaney-
chaos for discontinuous maps. Unlike the discrete-
ness of the space, the discontinuity of the map af-
fects Devaney’s chaos conditions rather unevenly. In
some cases, the conditions of Devaney-chaos are
not affected by the presence of the discontinuity.
For instance, it is known that a symmetric uniform
piecewise-affine elliptic rotation map

\[
f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y + 1 \\ x + 2y \cos \theta - \cos \theta \mod 1 \end{pmatrix},\]

(2.1)

which was originated, in part, from the information
overflow analysis problems in digital signal processing
[3, 6, 14, 17, 20, 22] is Devaney-chaotic in its aperi-
dodic set, if \(\theta = k\pi/4\) \((k = 1, 3) [12]\) or \(\theta = k\pi/5\)
\((k = 1, 2, 3, 4) [15]\). The periodic sets and the aperi-
dodic sets of the iterative dynamics given by the map
(2.1) for \(\theta = \pi/4\) and \(\theta = 2\pi/5\) are presented in
Figure 2.1 and Figure 2.2, respectively.

In other occasion, as we will see in the following
example, Definition 2.1 appears to be too inclusive.

Example 2.2. Let \(X \subset \mathbb{C}\) be the set given by
\(X = \{z : |z| < 1, \arg(z) = \theta \}\) for

Figure 2.1: \(\theta = \frac{\pi}{4}\). Figure 2.2: \(\theta = \frac{2\pi}{5}\).
The extreme case is the singular point (the black point in Figure 2.3) by drastically different places. The most cut off clarity depends on the initial condition around the singularity. The topological transitivity of such that \( \{ \phi(n) \} \) is dense in \([0, 1)\). Furthermore, \( f \) is Devaney-chaotic, according to Definition 2.1.

Proof. The map \( f : X \to X \) has two continuous pieces and one singular point. When \( 2k - 1 < \theta < 2k \) where \( k \in \mathbb{Z} \), we get \( z = (-1)\lceil \theta \rceil (1 - e^{-2\pi i (\theta - 1)\lceil \theta \rceil}) = -1 + e^{2\pi i \theta} \), which is the punctured circle \( \{ z \in \mathbb{C} : |z + 1| = 1, z \neq 0 \} \) (the left hind side circle of Figure 2.3). When \( 2k < \theta < 2k + 1 \) where \( k \in \mathbb{Z} \), on the other hand, we get \( z = (-1)\lceil \theta \rceil (1 - e^{-2\pi i (\theta - 1)\lceil \theta \rceil}) = 1 - e^{-2\pi i \theta} \), which yields \( \{ z \in \mathbb{C} : |z - 1| = 1, z \neq 0 \} \) (the right hind side circle of Figure 2.3). When \( \theta \in \mathbb{Z} \), we get the singular point, \( z = 0 \), which is doubly defined by \( \phi \).

Figure 2.4 and Figure 2.5 illustrate the dynamics of \( f \). Because of the periodicity of \( \phi \), we can decompose \( f \) by \( f = \phi \circ \sigma \), where \( \sigma(\theta) = \theta + \rho \) (mod 2), which is homeomorphic to the rotation map \( e^{2\pi i \theta} \to e^{i(\theta + \rho)} \). Because \( \rho \) is irrational, any \( \{ \sigma^n(\theta) : n \in \mathbb{N} \} \) is dense in \([0, 2)\). Taking a certain \( \theta_0 \) such that \( \{ \sigma^n(\theta_0) : n \in \mathbb{N} \} \cap \mathbb{Z} = \emptyset \), we get a point \( z_0 = \phi(\theta_0) \) such that \( \{ f^n(z_0) : n \in \mathbb{N} \} \) is dense in \( X \), and thus, the topological transitivity of \( f \) follows.

Figure 2.3 and Figure 2.5 depict the sensitive dependence on the initial condition around the singularity \( z = 0 \). A pair of points in the left hand side circle that are separated by the singularity are cut off and then sent to drastically different places. The most extreme case is the singular point (the black point in Figure 2.3), in which case, the point itself is separated and sent to two different positions (the black points in Figure 2.5). This observation applies to every pair of points. Because of the irrational rotation, every pair of points in either circle must be separated by the singularity \( z = 0 \) after a certain number of iterations of \( f \). In the next application of \( f \), they are separated by the singularity.

Recall that the map \( f \) in Example 2.2 has a singularity at \( z = 0 \), at which \( f \) is double-valued. Alternatively, we can regard \( f \) discontinuous but single valued at \( z = 0 \) by taking either one of the two branches. It is easy to see, however, these considerations make difference almost nowhere.

Calling \( f : X \to X \) in Example 2.2 chaotic, however, is somewhat problematic, in a sense that the dynamics is too trivial. In fact, if we identify the left hand side and the right hand side loops of Figure 2.3 by \(-1 + e^{2\pi i \theta} \equiv 1 - e^{-2\pi i \theta} \), then the quotient map \( f : X \to \tilde{X} \) is reduced to a rotation map \( \tilde{f} : [-1 + e^{2\pi i \theta}] \to [-1 + e^{2\pi i (\theta + \rho)}] \).

3 Singularities of Piecewise Isometric Dynamical Systems

A piecewise isometry is a discontinuous map on a union of finitely many convex regions (which we call,
atoms), in each of which the map is an isometry (which we call, the isometric component).

Piecewise isometries are important for our purpose because of the simplicity. That is, every non-trivial behavior (including chaotic behavior) of a piecewise isometric dynamics must come from the discontinuity, because the isometric components alone contribute nothing. In this paper, we restrict ourselves to the planar piecewise isometries, in which case each isometric component must be a composition of a rotation, a translation, and possibly an inversion.

The discontinuity takes place on the common boundaries of the atoms, but it is often more convenient to regard the map multiple valued there, as we did in Example 2.2. For this reason, we will use the term singularity instead of discontinuity from this point on.

Piecewise isometric dynamical systems often exhibit complex orbit behavior, some of which are visualized in Figure 1.1, Figure 2.1 and Figure 2.2. These apparent fractal patterns are generated by the singularity. The dark regions are given by the forward and the backward iterates of the singularity, which we call the singular set and denote $\Sigma$. Upon taking the closure ($\bar{\Sigma}$), we get the set of points whose orbits get arbitrarily close to the singularity, which we call the exceptional set. The interior of the white regions consists of the points whose orbits stay away from the singularity, and it is tedious but not difficult to prove that they turn out to be the periodic sets [11]. For this reason, we call the exceptional set $\bar{\Sigma}$ alternatively as the aperiodic set, even though it may contain some degenerate periodic points.1

Inside the periodic set, the dynamics of the piecewise isometric system is rather trivial. Unaffected by the singularity, the dynamics there must be an iteration of a composition of a translation, a rotation, and possibly an inversion. In the aperiodic set, on the other hand, the piecewise isometric dynamics can be quite complicated. In some cases, it is known that the piecewise isometric dynamical systems are Devaney-chaotic in its aperiodic sets [12, 15]. In other occasions, it is known that the aperiodic set consists of more than one invariant sets in which the dynamics is Devaney-chaotic [13]. Such invariant sets are called the chaotic sets. In many cases, the chaotic sets exhibit riddling [13]. Finally, it is also possible that the dynamics turns out to be too trivial to be called chaotic, as we saw in Example 2.2.

We aim to find a way to distinguish the trivial cases such as Example 2.2 from the non-trivial cases. We begin with investigating the behavior of the map on the singularity.

**Theorem 3.1.** Let $f : X \rightarrow X$ be a bounded invertible planar piecewise isometry. Let $f_1, \ldots, f_n$ be the isometric components of $f$ in the atoms $P_1, \ldots, P_n$ respectively. Suppose that $f$ has the cutting singularity on a curve segment $S_{ij} = \partial P_i \cap \partial P_j$, that is, $f_i(S_{ij}) \cap f_j(S_{ij})$ has length 0. Then, the quotient map $\hat{f}$ on the quotient space $\hat{X}$ given by the following identification is continuous in $P_i \cup P_j$.

$$p \equiv q \iff \begin{cases} 
\text{either } p = q; \\
\text{or } p = f_i(x), \; q = f_j(x), \;
\exists i, j \in \{1, \ldots, n\}, \; x \in S_{ij}; \\
\text{or } p = f_j(x), \; q = f_i(x), \;
\exists i, j \in \{1, \ldots, n\}, \; x \in S_{ij}.
\end{cases}$$

In other words, the discontinuity on $S_{ij}$ disappears.

**Proof.** See the author’s paper, [16].

We call the identification ($\equiv$) of Theorem 3.1, the patch-up identification, and the process of making $\hat{f} : \hat{X} \rightarrow \hat{X}$ out of $f : X \rightarrow X$, the isometric continuation. Some singularity completely disappears under the isometric continuation, but not in general. What it does, typically, is to postpone the singularity by one iteration.

Let us take Figure 3.1 and Figure 3.2 for examples. The piecewise isometry $f$ on the rhombus of Figure 3.1 consists of three isometric components, $f_{-1}, f_0, f_{+1}$, and three atoms, $P_{-1}, P_0, P_{+1}$. $f_0$ rotates $P_0$, $f_{+1}$ rotates $P_{+1}$ and then translates it upward, and $f_{-1}$ is applied similarly to $P_{-1}$ (Figure 3.2).

By taking the isometric continuation, which is the natural identification of the top edge $E_T$ and the bottom edge $E_B$ of the rhombus, we can remove the discontinuity at $\partial P_0 \cap \partial P_{+1}$ and at $\partial P_0 \cap \partial P_{-1}$. Moreover, applying the isometric continuation also to $f^{-1}$, we can identify the slanted edges as well, and consequently, we get $\hat{X} = \mathbb{T}^2$.

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1The degenerate periodic points, if they exist, must always be on $\Sigma$ [11]. For this reason, some people call $\Sigma \setminus \Sigma$ the aperiodic set. The trade off in this case is that $\Sigma \setminus \Sigma$ does not contain all the aperiodic points.
The singularity does not go away completely in general, however. As we can see from the splitting and the sliding of the arrow in Figure 3.1 and Figure 3.2, the cutting evolves to another type of singularity, which we call the **sliding singularity**. Note that the singularities presented in Figure 3.3. The first picture depicts the sliding singularity upon taking the isometric continuations. Consequently, the isometric continuation stops here. The sliding of the arrow in Figure 3.1 and Figure 3.2, generally, however. As we can see from the splitting and the sliding singularities, Besides the cutting and the sliding singularities, there is another type of singularity that we must consider, the **shuffling singularity**.

**Definition 3.2.** Let \( \{P_1, \ldots, P_n\}, \{f_1, \ldots, f_n\} \), \( X \) and \( f \) be as in Theorem 3.1. We say \( f \) has the **shuffling singularity** on a segment \( S_k \subset \partial P_1 \cap \partial P_j \), if there are a finite number of segments \( S_k \subset \partial P_1 \cap \partial P_j \), \( k \in \{1, 2, \ldots, r\} \) and a positive integer \( m \in \mathbb{Z}^+ \) that satisfy the following conditions.

1. For every \( S_k \), \( k \in \{1, 2, \ldots, r\} \), there exists a certain \( S_l \), \( l \in \{1, 2, \ldots, r\} \) such that \( \tilde{f}^m \circ f_{ik}(S_k) = \tilde{f}^m \circ f_{jl}(S_l) \) upon taking appropriate branches of the isometric continuation \( \tilde{f}^m \).
2. \( S_k \cap S_l \) has length 0, if \( k \neq l \).
3. Each \( S_k \) is the maximal segment with respect to the inclusion that satisfies (1) and (2).

Rather trivial example of a shuffling singularity is presented in Figure 3.3. The first picture depicts the atoms, while the second picture illustrates their images. Note that the singularities \( \partial P_i \cap \partial P_j \)'s are literally **shuffled** by the map.

In the next section, we will use these different types of singularities to re-define Devaney-chaos. The following theorem justifies this argument, by guaranteeing that there is no other types of singularities to consider.

**Theorem 3.3.** Let \( \{P_1, \ldots, P_n\}, \{f_1, \ldots, f_n\} \), \( X \) and \( f \) be as in Theorem 3.1, and let \( S \subset \partial P_i \cap \partial P_j \). Then, we must have one of the following.

1. For some \( m \in \mathbb{N} \), \( f^m \circ f_i = \tilde{f}^m \circ f_j \). That is, the singularity disappears after finitely many applications of the isometric continuation.
2. \( f \) has the shuffling singularity on \( S \).
3. \( f \) has the non-shuffling sliding singularity on \( S \).

### 4 Conclusion: Devaney-chaos for Discontinuous Dynamical Systems

The key idea behind this project was the realization that the shuffling singularity is non-chaotic.

**Theorem 4.1.** Let \( \{P_1, \ldots, P_n\}, \{f_1, \ldots, f_n\} \), \( X \) and \( f \) be as in Theorem 3.1. Let \( S_1 \subset \partial P_1 \cap \partial P_j \), \( \cdots \), \( S_r \subset \partial P_i \cap \partial P_j \), and \( m \in \mathbb{N} \) be as in Definition 3.2. Let \( x_k \) and \( y_k \) be the points sufficiently close to \( S_k \) where \( k \in \{1, \ldots, r\} \), such that each \( x_k \) and \( y_k \) is positioned in the same geometrical location from \( S_k \) as \( x_1 \) and \( y_1 \) are from \( S_1 \). Suppose further that each \( x_k \) and \( y_k \) separated up to \( m + 1 \) iterations only by \( S_k \), where the crossing point is \( c_k = x_k \cup y_k \cap S_k \). Then, for every \( k \in \{1, \ldots, r\} \), there exists a unique \( l \in \{1, \ldots, r\} \) that satisfies the following conditions.

1. \( \tilde{f}^m \circ f_{ik}(c_k) = \tilde{f}^m \circ f_{jl}(c_l) \). That is, the two points merge to the same point after \( m + 1 \) iterations, upon taking appropriate branches.
2. The curve segment \( \tilde{f}^{m+1}(x_{k})\tilde{f}^{m+1}(y_{k}) \) is a line segment that runs through the (common) point \( \tilde{f}^m \circ f_{ik}(c_k) = \tilde{f}^m \circ f_{jl}(c_l) \).
3. \( \mu_1(x_k y_k) = \mu_1 \left( \tilde{f}^{m+1}(x_k)\tilde{f}^{m+1}(y_k) \right) \). That is, the length of the line segment is preserved.

**Proof.** See [16]

**Remark 4.2.** Theorem 4.1 tells us that the shuffling singularity shuffles not only the curve segments of the singular set, but also their one-sided neighborhoods as well. Consequently, the piecewise isometric dynamics around the shuffling singularity merely shuffles the \( x_k \)'s and \( y_k \)'s. Hence, even though \( d(f(x_k), f(y_k)) \) suddenly gets large, the minimal distance between all of them stays constant. That is, \( \min\{d(f(x_k), f(y_l)) : 0 \leq k, l \leq r \} = \min\{d(x_k, y_l) : 0 \leq k, l \leq r \} \). Consequently, the shuffling singularity has no sensitive dependence upon the initial condition in this sense.

Figure 3.3 illustrates the idea behind Remark 4.2. Here, the points \( x_k \)'s and \( y_k \)'s are represented as the
dots near the singular edges.

Remark 4.2 allows us to draw the following conclusion, the adjusted definition of Devaney-chaos that can be used to the discontinuous dynamical systems as well. Theorem 4.1 and Remark 4.2 justifies the adjustment and Theorem 3.3 guarantees that this improvement is as strong as possible.

Definition 4.3. Let \((X, d)\) be a metric space. Then, a map \(f : X \rightarrow X\) is said to be Devaney-chaotic on \(X\) if it satisfies the following conditions.

1. \(f\) has sensitive dependence on initial conditions. That is, there exists a certain \(\delta > 0\) such that, for any \(x_0 \in X\) and \(\epsilon > 0\), there exist some \(y_0 \in X\) where \(d(x_0, y_0) < \epsilon\), such that for any pair of finite sets \(\{x_0, \ldots, x_r\}, \{y_0, \ldots, y_r\} \subseteq X\), \(\min\{d(f^m(x_k), f^m(y_l)) : 0 \leq k, l \leq r\} > \delta\), for some \(m \in \mathbb{N}\).

2. \(f\) is topologically transitive (Definition 2.1).

References:


4The light dots outside the strip represent the copies of the points in the other side of the strip. Recall the the outer edges can be identified through the isometric continuation.