1 Introduction

In this paper we will consider the fifth order dispersive equations:

\[ u_t + au_{xxxxx} = 0, \quad 0 \leq t \leq T \quad (1.1) \]

with initial and periodic boundary value:

\[
\begin{align*}
&u(x, 0) = f(x), \\
&u(x, t) = u(x + L, t). \quad (1.2)
\end{align*}
\]

In scientific and engineering computing, we need to solve large equation set by numerical methods. Considering the stability and accuracy of explicit schemes and the computation difficulty of implicit methods, that is, simple for computation and good stability. Many numerical methods have been established for third order dispersive equations [1-4], But researches on high order dispersive equations have been scarcely presented. Recently with the development of parallel computer many scientists pay much attention to solving use of asymmetry schemes at adherent grid points can be denoted explicitly. Furthermore, by alternating use of asymmetry schemes at adherent grid points and different time levels, the AGE method can lead to the property of unconditional stability. But the original AGE method has only two order accurate for spatial step. The AGE method is soon applied to convection-diffusion equations and hyperbolic equations [6,7]. In [8-11], AGE method is applied to solve semi-linear and non-linear equations. Several AGE methods are given for two-point linear and non-linear boundary value problems in [12-13]. To our knowledge AGE methods for fifth order dispersive equations have scarcely been presented.

We will organize this paper as follows: In section 2, we present a group of asymmetric schemes. Based on the schemes a class of unconditionally stable alternating group explicit finite difference method will be derived. Stability analysis for the alternating group method is given in section 3. In section 4, Results of numerical experiments on stability and accuracy are presented.

2 The Alternating Group Method

The domain \( \Omega : [0, L] \times [0, T] \) will be divided into \((m \times N)\) meshes with spatial step size \( h = \frac{1}{m} \) in x direction and the time step size \( \tau = \frac{T}{N} \). Grid points are denoted by \((x_i, t_n)\) or \((i, n)\), \(x_i = ih(i = 0, 1, \cdots n, m), t_n = n\tau(n = 0, 1, \cdots N)\). The numerical solution of (1.1) is denoted by \(u^n_i\), while the exact solution \(u(x_i, t_n)\). Let \(r = \frac{\tau}{2h^2}\).

We first present twelve saul’yev asymmetry schemes to approach (1.1) at \((i, n + \frac{1}{2})\) as follows:

\[
(1 - 2r)u_{i+1}^{n+1} + 5ru_{i+2}^{n+1} - 4ru_{i+3}^{n+1} + ru_{i+4}^{n+1} = 2ru_{i-3}^{n-3}
\]

\[
-8ru_{i-2}^{n-2} + 10ru_{i-1}^{n} + (1 - 2r)u_{i}^{n} - 5ru_{i+1}^{n} + 4ru_{i+2}^{n} - ru_{i+3}^{n-3} \quad (2.1)
\]

\[
-2ru_{i-1}^{n-1} + ru_{i}^{n} + 5ru_{i+1}^{n} - 4ru_{i+2}^{n} + ru_{i+3}^{n+1} = 2ru_{i-3}^{n-3}
\]

\[
-8ru_{i-2}^{n-2} + 8ru_{i-1}^{n} + u_{i}^{n} - 5ru_{i+1}^{n} + 4ru_{i+2}^{n} - ru_{i+3}^{n+1} \quad (2.2)
\]
\[
3ru_{i-1}^{n+1} - 5ru_{i-2}^{n+1} + u_{i}^{n+1} + 5ru_{i+1}^{n+1} - 4ru_{i+2}^{n+1} + ru_{i+3}^{n+1} = 2ru_{i-3} - 5ru_{i-2} + 5ru_{i-1}^{n+1} + u_{i}^{n+1} - 5ru_{i+1}^{n+1} + 4ru_{i+2}^{n+1} - ru_{i+3}^{n+1}
\]

(2.3)

\[
-ru_{i-3}^{n+1} + 4ru_{i-2}^{n+1} - 5ru_{i-1}^{n+1} + u_{i}^{n+1} + 5ru_{i+1}^{n+1} - 5ru_{i+2}^{n+1} + 2ru_{i+3}^{n+1} = ru_{i-3} - 4ru_{i-2}^{n+1} + 5ru_{i-1}^{n+1} + u_{i}^{n+1} - 5ru_{i+1}^{n+1} + 3ru_{i+2}^{n+1} + 3ru_{i+3}^{n+1}
\]

(2.4)

Using the schemes mentioned above, we will have three basic independent computation groups:

1) \(\kappa_1\) group: twelve grid points are involved, and (2.1)–(2.12) are used at each grid point respectively.

2) \(\kappa_2\) group: six inner points are involved, and (2.1)–(2.6) are used respectively.

3) \(\kappa_3\) group: six inner points are involved, and (2.7)–(2.12) are used respectively.

Based on the basic point groups above, we construct the alternating group method in two cases as follows:

Case 1: Let \(m = 12s\), where \(s\) is an integer. First at the \((n+1)\)-th time level, we divide all the \(m\) grid points into \(s\) \(\kappa_1\) groups. Twelve grid points are included in each group, named \((i + k, n + 1), \ k = 0, 1, \ldots, 11\), and (2.1)-(2.12) are applied respectively. Second at the \((n+2)\)-th time level, we divide all the \(m\) grid points into \((s + 1)\) groups. \(\kappa_3\) group is used to get the solution of the left six grid points \(u_1^{n+2}, u_2^{n+2}, u_3^{n+2}, u_4^{n+2}, u_5^{n+2}, u_6^{n+2}\). \(\kappa_1\) group is used in each of the following 6 point groups, while \(\kappa_2\) group is used in the right six grid points \(u_{m-1}, u_{m-2}, u_{m-3}, u_{m-4}, u_{m-5}\).

It is obvious that computation in the whole domain can be fulfilled in many sub domains independently, and the basic computation groups above are properly used in each sub domain. So the alternating group method has the property of parallelism.

Let \(U^n = (u_1^n, u_2^n, \ldots, u_m^n)^T\), then we can denote the alternating group method I as follows:

\[
(I + rH_1)U^{n+1} = (I - rH_2)U^n
\]

(2.13)

\[
H_1 = \begin{pmatrix}
H_{11} & \cdots & H_{11} \\
H_{11} & \cdots & H_{11} \\
H_{11} & \cdots & H_{11}
\end{pmatrix}_{m \times m}
\]

\[
H_2 = \begin{pmatrix}
H_{21} & \tilde{Q} \\
H_{21} & \tilde{Q} \\
\cdots & \cdots
\end{pmatrix}_{m \times m}, \quad \tilde{Q} = \begin{pmatrix}
H_{11} \\
H_{11} \\
\cdots
\end{pmatrix}
\]

\[
H_{11} = \begin{pmatrix}
H_{111} & H_{112} \\
H_{113} & H_{114}
\end{pmatrix}
\]
we divide all the \( m \) grid points into \((s + 1)\) groups. "\( \kappa_1 \)" group is used in each of the left \( s \) groups, while "\( \kappa_2 \)" group is used in the right six grid points \( u_{n+2}^{m+2}, u_{n+1}^{m+2}, u_{n}^{m+2}, u_{n-1}^{m+2}, u_{n}^{m+1}, u_{n}^{m} \). Second at the \((n+2)\)-th time level, we still divide all the \( m \) grid points into \((s + 1)\) groups. "\( \kappa_3 \)" group is used to get the solution of the left six grid points \( u_{n+1}^{t}, u_{n+2}^{2}, u_{n+3}^{2}, u_{n+2}^{3}, u_{n+2}^{4}, u_{n+2}^{5} \), while "\( \kappa_1 \)" group is used in each of the following \( s \) point groups.

We denote the alternating group method II as follows:

\[
\begin{align*}
(I + r \tilde{H}_1)U_{n+1}^{t} &= (I - r \tilde{H}_2)U_n^t \\
(I + r \tilde{H}_2)U_{n+2}^{t} &= (I - r \tilde{H}_1)U_{n+1}^{t}
\end{align*}
\]

\[
\tilde{H}_1 = \begin{pmatrix}
H_{11} & & \\
& \ddots & \\
& & H_{11}
\end{pmatrix}_{m \times m},
\]

\[
\tilde{H}_2 = \begin{pmatrix}
H_{21} & & \\
& \ddots & \\
& & H_{11}
\end{pmatrix}_{m \times m},
\]

The alternating use of the asymmetry schemes (2.1)-(2.12) can lead to partly counteracting of truncation error, and then can increase the numerical accuracy.
3 Stability Analysis

Kellogg Lemma\textsuperscript{[14]} Let $r > 0$, and $G + GT$ is non-negative definite real matrix, then:

\[
\begin{align*}
\| (I + rG)^{-1}\|_2 & \leq 1 \\
\| (I - rG)(I + rG)^{-1}\|_2 & \leq 1
\end{align*}
\] (3.1)

Theorem 1 The alternating group method I denoted by (2.13) is unconditionally stable.

Proof: Obviously $H_1 + H_1^T$ and $H_2 + H_2^T$ are both nonnegative definite real matrices. Then we have:

\[
\begin{align*}
\| (I + rH_1)^{-1}\|_2 & \leq 1, \| (I - rH_1)(I + rH_1)^{-1}\|_2 \leq 1 \\
\| (I + rH_2)^{-1}\|_2 & \leq 1, \| (I - rH_2)(I + rH_2)^{-1}\|_2 \leq 1.
\end{align*}
\]

Let $n$ be an even number. From (2.13) we have

\[
U^n = HU^{n-2}
\]

here

\[
H = (I + rH_2)^{-1}(I - rH_1)(I + rH_1)^{-1}(I - rH_2).
\]

Let $H = (I + rH_2)^{-1}(I - rH_1)(I + rH_1)^{-1}(I - rH_2)$, then we have $\rho(H) = \rho(H) \leq \|H\|_2 \leq 1$, which shows the alternating group method I given by (2.13) is unconditionally stable. So theorem 1 is proved.

Analogously we have:

Theorem 2 The alternating group method II denoted by (2.14) is also unconditionally stable.

4 Numerical Experiments

Let $a = 1$, $L = 2.0$, $u(x, 0) = \cos(\pi x)$, then the exact solution of (1.1) is denoted as below:

\[
u(x, t) = \cos(\pi x - \pi^5 t)
\]

Let $A.E. = |u^n - u(x_i, t_n)|$, $P.E. = \frac{|u^n - u(x_i, t_n)|}{u(x_i, t_n)}$ denote maximum absolute error and relevant error respectively. We compare the numerical results of the presented alternating group method in this paper with the results of the full implicit Crank-Nicolson scheme (IC-N) as follows:

Table 1: results of comparison $m = 96$, $r = 0.2$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$A.E.$</th>
<th>$A.E.\text{ IC-N}$</th>
<th>$P.E.$</th>
<th>$P.E.\text{ IC-N}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1000\tau$</td>
<td>$3.618 \times 10^{-6}$</td>
<td>$3.462 \times 10^{-6}$</td>
<td>$5.453 \times 10^{-3}$</td>
<td>$5.319 \times 10^{-3}$</td>
</tr>
<tr>
<td>$3000\tau$</td>
<td>$1.122 \times 10^{-5}$</td>
<td>$1.030 \times 10^{-5}$</td>
<td>$1.813 \times 10^{-2}$</td>
<td>$1.749 \times 10^{-2}$</td>
</tr>
<tr>
<td>$7000\tau$</td>
<td>$2.623 \times 10^{-5}$</td>
<td>$2.405 \times 10^{-5}$</td>
<td>$5.004 \times 10^{-2}$</td>
<td>$4.923 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 2: results of comparison $m = 120$, $r = 2$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$A.E.$</th>
<th>$A.E.\text{ IC-N}$</th>
<th>$P.E.$</th>
<th>$P.E.\text{ IC-N}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1000\tau$</td>
<td>$2.942 \times 10^{-6}$</td>
<td>$2.918 \times 10^{-6}$</td>
<td>$5.717 \times 10^{-3}$</td>
<td>$5.706 \times 10^{-3}$</td>
</tr>
<tr>
<td>$3000\tau$</td>
<td>$8.713 \times 10^{-6}$</td>
<td>$8.651 \times 10^{-6}$</td>
<td>$2.014 \times 10^{-2}$</td>
<td>$1.988 \times 10^{-2}$</td>
</tr>
<tr>
<td>$7000\tau$</td>
<td>$2.085 \times 10^{-5}$</td>
<td>$2.017 \times 10^{-5}$</td>
<td>$6.647 \times 10^{-2}$</td>
<td>$6.616 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 3: results of comparison $m = 120$, $r = 10$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$A.E.$</th>
<th>$A.E.\text{ IC-N}$</th>
<th>$P.E.$</th>
<th>$P.E.\text{ IC-N}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1000\tau$</td>
<td>$1.454 \times 10^{-5}$</td>
<td>$1.443 \times 10^{-5}$</td>
<td>$3.933 \times 10^{-2}$</td>
<td>$3.915 \times 10^{-2}$</td>
</tr>
<tr>
<td>$3000\tau$</td>
<td>$4.362 \times 10^{-5}$</td>
<td>$4.317 \times 10^{-5}$</td>
<td>$8.412 \times 10^{-1}$</td>
<td>$8.393 \times 10^{-1}$</td>
</tr>
<tr>
<td>$7000\tau$</td>
<td>$1.024 \times 10^{-4}$</td>
<td>$1.006 \times 10^{-4}$</td>
<td>$1.854$</td>
<td>$1.838$</td>
</tr>
</tbody>
</table>

Results of table 1-4 show that the present AG method is stable even in large $r$, and is of nearly the same accuracy as the full implicit Crank-Nicolson scheme. With the increase of grid points, we can obtain higher accuracy.

5 Conclusions

In this paper, we present a class of alternating group method for fifth order dispersive equations, which is of intrinsic parallelism, and verified to be unconditionally stable. Numerical results show that the method is of high accuracy. Considering the absolute stability of the method, it doesn’t lead to numerical vibration in computation. Based on the characters, it is an effective method in solving large equation set.

References:


