Projection-iterative methods for solving Singular Integral Equations in Lebesgue spaces

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Abstract: This article is devoted for elaboration of Projection-iterative algorithms for approximate solution of Singular Integral Equations based on the collocation method. We study the case when the equations are defined on the arbitrary smooth closed contours of complex plane. Theoretical background was obtained in Lebesgue spaces.

Key-words: Lebesgue Spaces, Iterative Projection Methods, Lagrange Polynomial

1 Introduction

The large range of applied problems of mathematical physics, the theory of elasticity, thermoelasticity and others [1], [2] leads to solving the linear SIE and their systems with Cauchy kernel. It is known that the solution of SIE in the closed form is possible only in some particularly cases and even in these cases the receipt of the numerical results evoke frequently considerable difficulties. This circumstances cause the large interest for elaboration and foundation of methods for approximate solution of different types of SIE. At the present time the theory of projection methods for solving the SIE on different contours in different functional spaces is sufficiently well elaborated with large assumptions relatively their coefficients, regular kernels and right part. The projection methods for approximate solution of SIE processing the simple computation schemes, have one essential deficiencies. The process of approximate solving SIE with given high order accuracy by means of projection methods leads to necessity of solving the systems of linear algebraic equations (SLAE) of high orders with ill-conditioned matrices. This fact does not give an opportunity to apply the projection methods in effective way for obtaining with high accuracy the approximate solution of SIE because the round-off errors in solving the SLAE lead to big errors in final result. But the iterative methods for solving SIE do not posses such a deficiency.

We did not study the SIE in space of continuous functions because the singular operator is unbounded. The main task of this article is theoretical background of computational schemes of diverse iterative methods for solving the normal case of SIE with Cauchy kernels, on the arbitrary smooth closed contours in Lebesgue space. The Lagrange operator is unbounded in Lebesgue spaces.

2 Theoretical preparation

Let \( \Gamma \) be a smooth Jordan border which divides the complex plane \( \mathbb{C} \) into two parts, \( F^- \) and \( F^+ \). Assume that \( F^- \) contains the infinity and \( F^+ \) contains the origin. Therefore \( t = 0, F^- = C \setminus \{ F^+ \cup \Gamma \} \). Let \( z = \psi(w) \) be a analytic function that maps conformably \( \Gamma_0 = \{|w| > 1\} \) on the surface \( \Gamma \) so that \( \psi(\infty) = \infty, \psi^{(0)}(\infty) > 0 \).

We shall assume that the function \( z = \psi(w) \) has second derivative , satisfying on \( \Gamma_0 \) the Hölder condition with some parameter \( \nu \) \((0 < \nu < 1)\); the class of such contours is denoted [3] by \( C(2; \nu) \).

Let \( \{t_j\}_{j=0}^{2n} \) \( (n \text{ is natural}) \) be a set of distinct points on \( \Gamma \). By \( U_n \) we denote the operator which
maps any function \( g(t) \) defined on \( \Gamma \) into its interpolating polynomial defined by using the points \( t_j \):

\[
(U_n g)(t) = \sum_{j=0}^{2n} g(t_j) \cdot l_j(t), \quad t \in \Gamma,
\]

where

\[
l_j(t) = \left( \frac{t_j}{t} \right)^n \prod_{k=0, k \neq j}^{2n} \frac{t - t_k}{t_j - t_k} \equiv \sum_{k=-n}^{n} \Lambda_k^{(j)} t_k, \quad t \in \Gamma. \tag{1}
\]

3 The approximation of functions in Lebesgue spaces

Let \( L_p(\Gamma)(1 < p < \infty) \) be complex space of the functions \( g(t) \in L_p(\Gamma) \) with the norm

\[
||g|| = \left( \frac{1}{l} \int_{\Gamma} |g|^p |d\tau| \right)^{\frac{1}{p}}, \tag{2}
\]

where \( l \) is the length of \( \Gamma \).

**Theorem 1** Let

\[
t_j = \psi(w_j) = \psi(\exp \left( \frac{2\pi i}{2n+1} (j-n) \right)), \quad j = 0, 2n.
\]

then \( ||U_n||_{\infty \rightarrow L_p} \leq m_1(< \infty) \), where \( m_1 = m_1(p) \) is a constant which depends from \( p \).

**Remark** Previous theorem prove that the operator \( U_n \) which mapping from \( C(\Gamma) \) in \( L_p(\Gamma) \) is bounded

In [3] is proven that operator \( U_n : L_p(\Gamma) \rightarrow L_p(\Gamma) \) is unbounded even in case \( \Gamma = \Gamma_0 \) is a unit circle with center in point zero

\[
||U_n||_{L_p} = \infty \quad 1 < p < \infty.
\]

**Theorem 2** Let \( \Gamma \in C(2, \mu) \) and \( t_j (j = 0, 2n) \) is Fejer points (3) on \( \Gamma \). Then for all continuous function \( g(t) \) on \( \Gamma \) the following inequality takes place

\[
||g - U_n g||_p \leq (1 + m_1(p)) E_n(g; \Gamma). \tag{4}
\]

where \( E_n(g; \Gamma) \) is the best approximation of function \( g(t) \) with polynomials (1)

\[
E_n(g) = \inf_{p_n \in P_n} ||g - p_n||; \quad P_n = \{ \sum_{k=-n}^{n} \beta_k t^k \}
\]

**Remark** If \( g(t) \in H^{r;\alpha}(\Gamma), \quad r = 0, 1, 2, \ldots \), then

\[
||g - U_n g||_p \leq (1 + m_1(p)) c_\alpha^r \frac{H(g^{(r)}; \alpha)}{n^{r+\alpha}}. \tag{5}
\]

The proofs of theorems 1,2 can be found in [3].

4 Numerical schemes

We develop the projection-iterative method for approximate solution of SIE. We study the following SIE

\[
(V \varphi) = a(t) \varphi(t) + \frac{b(t)}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{\Gamma} k(t, \tau) d\tau = f(t), \quad t \in \Gamma \tag{6}
\]

on the smooth closed contour \( \Gamma \in C(2, \mu) \), where \( a(t), b(t), k(t, \tau), f(t) \) are given functions and \( \varphi(t) \) is an unknown function. According to the projection-methods the initial approximation \( \varphi_0(t) \) is an approximate solution of the equation (6) obtained by collocation method

\[
\varphi_0(t) = \sum_{\nu=-n}^{n} \alpha_\nu t^\nu,
\]

where \( \alpha_\nu \) are unknown constants determined in the collocation method from the system of linear algebraic equation (SLAE)

\[
\sum_{\nu=-n}^{n} c(t_j) t^{\nu}_j \alpha_\nu + \sum_{\nu=-n}^{n} t^{\nu}_j \alpha_\nu + \sum_{\nu=-n}^{n} \frac{1}{2\pi i} \int_{\Gamma} k(t_j, \tau) \tau^\nu d\tau \alpha_\nu
\]
where the points $t_j$ are calculated by (3), $c(t) = a(t) + b(t)$, $d(t) = a(t) - b(t)$. If

$$y_k(t) = \varphi_{k-1}(t) + (R[f - K\varphi_{k-1}])(t),$$

$$k = 1, 2, \ldots, (7)$$

where $R$ is the regularization of the operator $K$:

$$\begin{align*}
(Rx)(t) & = \frac{A(t)}{a^2(t) - b^2(t)}x(t) - \frac{b(t)}{a^2(t) - b^2(t)} \frac{1}{\pi i} \int_{\Gamma} \frac{x(\tau)}{\tau - t} d\tau, \\
& = \sum_{k=1}^{n} c^k(t) t^k, k = 1, 2, \ldots, (8)
\end{align*}$$

then we seek the elements of corrections in the form

$$\omega_k(t) = \sum_{\nu=-n}^{n} c^k_{\nu} t^{\nu}, k = 1, 2, \ldots, (9)$$

where the unknown constants $c^k_{\nu}$ are determined from the SLAE

$$\begin{align*}
(U_n K \omega_k)(t) & = \\
(U_n R[a - K\varphi_{k-1}])(t), \\
k = 1, 2, \ldots, (10)
\end{align*}$$

$U_n$ being the Lagrange operator by points (3). Then the following iteration are determined as follows:

$$\begin{align*}
\varphi_k(t) & = y_k(t) + \\
\omega_k(t) + (R\omega_k)(t), k = 1, 2, \ldots, (11)
\end{align*}$$

5 Main theorem

Theorem 3 Let

- $\Gamma \in C(2; \mu)$;
- $a(t), b(t), \text{ belong to } H\alpha(\Gamma); 0 < \alpha < 1$;
- $c(t) = a(t) + b(t) \neq 0, d(t) = a(t) - b(t) \neq 0$;
- $\text{ind } (c(t)/d(t))=0$;
- $k(t, \tau) \in C(\Gamma), f(t) \in C(\Gamma)$;
- $\text{dim Ker } V \neq 0$;
- $t_j(t)$ are calculated by (3).

The SIE has an unique solution for any right part $f(t)$ in $C(\Gamma)$. Then the iterative process (7)-(11) with the regularization $R$ from the SIE (6)

$$\begin{align*}
(Rx)(t) & = [a(t) + b(t)]^{-1} \\
& = \left\{ \begin{array}{ll}
1/2[E + N^{-1}(t)]\kappa(t) + \\
1/2[E - N^{-1}(t)] \int_{\Gamma} \frac{x(\tau)}{\tau - t} d\tau.
\end{array} \right.
\end{align*}$$

where $E$ is the identity matrix $N(t) = [a(t) - b(t)] \times [a(t) + b(t)]^{-1}$ and the approximate solutions $\varphi_k(t)$ converge as $k \to \infty$ to the exact solution $\varphi^*(t)$ of the system of SIE (6) with the rate

$$||\varphi^* - \varphi_k||_{L_p} \leq \left[ O \left( 1/n^{\alpha} \right) + O \left( \omega^{k} \left( k; \frac{1}{n} \right) \right) \right]^{k} + O \left( \omega \left( f; \frac{1}{n} \right) \right),$$

where $\omega(\cdot; \delta)$ is the modulus of $(\cdot)$.

The proof of this theorem is similar as in [4].

References: