Sampling Rates for the First-Order Sampling of Two-Band Signals

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Abstract: - Close-form representations for the permissible sampling rates on the first-order sampling of two-band bandpass signals are presented in this paper. It is shown that the new sampling rates are much lower than the Nyquist rate for signals with scattered passbands. A fast algorithm is developed for the symbolic operations on intervals, which makes the proposed sampling scheme more applicable.

Key-Words: First-order sampling, Nyquist rate, Permissible sampling rates, Aliasing, Ideal bandpass filter, *Csinc*-interpolation

1 Introduction

Digital Signal Processing (DSP) has been the driving force for the development of modern technologies in communication. Its applications have been seen from space technology to household electronics. An indispensable component of digital signal processing is sampling because an analog signal has to be sampled before any DSP procedures can be applied.

The process of selecting values of an analog signal at discrete-time instants is called sampling. A sampling device is used to take measurements of the analog signal at a regular interval of time. The sampling interval, the reciprocal of which is called sampling rate, has to be carefully selected so that the samples capture the characteristics of the original analog signal. A well-known industrial standard for sufficient sampling of an analog signal is that the sampling rate is at least twice the highest frequency, also known as the Nyquist rate [J. G. Proakis and D. G. Manolakis, 1996], of the signal. The challenge of effective sampling comes from signals with high frequency components because the sampling device has to perform at a much higher rate in order to cope with the high frequencies of the signal, and such sampling devices are expensive to make. As the modern technologies advance at a faster pace than ever, DSP technologies have to be at least one-step ahead. Particularly with the emergence of broad-band signals from space, commercial applications, and military, the highest frequency of which is usually at the range of mega-hertz or even giga-hertz, construction of advanced sampling mechanisms have become one of the most active research areas in digital signal processing. The goal of developing new sampling methods is always the same, that is, to reduce the sampling rate to a level that is much lower than the Nyquist rate.

In the past decade or so, higher-order sampling methods were developed to lower the sampling rate [Moon, 2000, Mitra, *et. al*., 1993, Xiao, 1995]. The idea of using a guard-band to reduce the susceptibility of the permissible sampling rates to aliasing was introduced in [Vaughan, *et. al*., 1991]. Similar treatment of the problem can be found in [Gaskell, 1978; Gregg, 1979; Coulson, 1994]. However, due to the excessive structural complexity, the implementation of those higher-order methods may add more cost to the making of such sampling devices.

It is observed that most broad-band signals are passband signals with existence of significant gaps among the spectral components of the signal displayed from the frequency domain [Proakis and Manolakis, 1996], see Fig.1.1. The objective of this work is to develop low-cost sampling methods to achieve a lower sampling rate than the Nyquist rate by utilizing the gaps among the spectra. The proposed sampling mechanism is guaranteed to be low-cost because only first-order sampling is considered, which has the simplest design structure, known as sample-and-hold. The idea of deriving such new sampling rates, both optimal and admissible sampling rates, is that the spectra of a passband signal do not intersect with each other (anti-aliasing) as they shift horizontally at a step size that is equal to the designated sampling rate.

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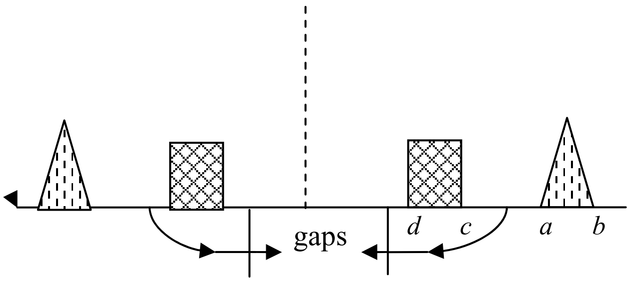

Fig 1 Two-band passband signals with gaps

Introduce the following interval notation for the compact support of a two-band signal in the frequency domain

$$I_{[d,c]\cup[a,b]} = [-c,-d]\cup[-b,-a]\cup[d,c]\cup[a,b] \quad (1)$$

A signal $f(t)$ is said to be bandpassed to $[d,c]\cup[a,b]$, see Fig. 1.2, if its Fourier transform $F(\omega)$ satisfies the condition $F(\omega)=0$ if $\omega \notin I_{[d,c]\cup[a,b]}$.

The illustration in Fig 1.2 gives a general idea of a signal bandpassed to $[d,c]\cup[a,b]$. Two bands on the negative frequency axis are mirror images of the original two bands on the positive side. In this setting, there are six possible pairings to consider for the interaction between two bands, in which two pairs are redundant. In section 2, we present results on the feasible step sizes for each pair and design a fast algorithm to calculate the feasible sampling rates from the six sets of step sizes.

2 Two-band sampling algorithm

We begin with introducing the symbols used in the derivation of feasible step sizes for the two-band signals.

Let the expression $I^{m\Delta}_{[a,b]} = [a+m\Delta, b+m\Delta]$ represent shifted version of the closed interval $I_{[a,b]}=[a,b]$, where $m \in \mathrm{N}$,and $\Delta \in R^+$. Throughout the text, it is always assumed that the intervals $[a,b]$ and $[d,c]$ are closed, disjoint, and satisfy $d<c<a<b$. The symbol Δ_f will be used to designate a feasible step size. Δ_f is a feasible step size for two intervals $[a,b]$ and $[d,c]$ if the following conditions are satisfied,

i.) $I^{m\Delta}_{[a,b]}$ intersects $I^{n\Delta}_{[d,c]}$ at only one point, or the intersection is empty, for all non-negative integers m and n.

ii.) $I^{m\Delta}_{[a,b]}$ intersects $I^{n\Delta}_{[a,b]}$ at only one point, or the intersection is empty, for all non-negative integers m and n.

iii.) $I^{m\Delta}_{[d,c]}$ intersects $I^{n\Delta}_{[d,c]}$ at only one point, or the intersection is empty, for all non-negative integers m and n.

We first present four lemmas, including the closed form formula for the feasible step sizes for two-band. The proofs of the first three lemmas are left out because they are straightforward. These lemmas will be used in the proof of the main theorem.

Lemma 2.1 Let $[d,c]$ and $[a,b]$ be closed disjoint intervals such that $\Delta \ge (c-d)$ and $\Delta \ge (b-a)$. If $\exists$ $m \in \mathrm{Z}^+$ such that $c+m\Delta \le a$ and $d+(m+1)\Delta \ge b$, then Δ is a feasible step size.

Introducing an integer

$$t = \left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor, \quad (2)$$

where $\lfloor\ \rfloor$ is the floor function, for example, $\lfloor 2.89 \rfloor = 2$. This value can be thought of as the capacity of the gap between the two intervals $[a,b]$ and $[d,c]$ to accommodate shifting the interval through the gap in between without causing intersections. Using this value, we can partition the positive side of the real line and determine feasibility of the step size.

Lemma 2.2 If $t = \left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor$, then $\frac{b-d}{t+1} \le \frac{a-c}{t}$.

Lemma 2.3 Let $t = \left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor$; then $\frac{b-d}{t+1} \ge (b-a)$ and $\frac{b-d}{t+1} \ge (c-d)$.

Theorem 2.1 Let $t = \left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor$ and

$$\Delta_f \in \bigcup_{n=1}^{t-1}\left[\frac{b-d}{t-n+1}, \frac{a-c}{t-n}\right] \bigcup [b-d, \infty), \quad (3)$$

then Δ_f is a feasible step size.

Proof: If $\Delta_f \in [b-d,\infty)$, then it is greater than the Nyquist rate, and is feasible. Thus, we only consider $\Delta_f \in \bigcup_{n=1}^{t-1}\left[\frac{b-d}{t-n+1}, \frac{a-c}{t-n}\right]$. Then $\Delta_f \in \left[\frac{b-d}{t-k+1}, \frac{a-c}{t-k}\right]$ for some $k=1,2,\ldots,t-1$, implying $\frac{b-d}{t-k+1} \le \Delta_f \le \frac{a-c}{t-k}$. Thus, one has

$$c+(t-k)\,\Delta_f \le a$$

and

$$d+(t-k+1)\,\Delta_f \ge b\,.$$

From Lemmas 2.1 and 2.3, Δ_f is a feasible solution with $\frac{b-d}{t+1}$ being the smallest feasible solution and hence the optimal solution.

It is given explicitly in Theorem 2.1 all possible step sizes that are less than the Nyquist rate, if any, that forbid intersections between the bands (intervals) during shifting along the frequency axis. For the two band case, see Fig. 1.1, there are actually four bands in existence. Hence, there are six possible pairings among the four bands. With the help of Theorem 2.1, one can calculate the intervals for each pair of intervals, then, find the intersection of all the intervals, which leads to the final solution for the feasible step sizes for the direct sampling of two-band passband signals.

Introduce the ideal bandpass filters In that case

$$S(\omega)=\begin{cases}1 & \omega\in I_{[d,c]\cup[a,b]}\\ 0 & otherwise\end{cases} \qquad (4)$$

The inverse Fourier Transform of $S(\omega)$ is

$$s(t)=\frac{\sigma_1}{\pi}c\sin c_{[\omega_1,\sigma_1]}(t)+\frac{\sigma_2}{\pi}c\sin c_{[\omega_2,\sigma_2]}(t) \quad (5)$$

where

$$\mathrm{csinc}_{[\omega,\sigma]}(t)=\frac{\cos(\omega t)\sin(\sigma t/2)}{\sigma t/2},$$

$$\sigma_1=c-d,\ \omega_1=\frac{c+d}{2},\ \sigma_2=b-a,\text{ and } \omega_2=\frac{a+b}{2}.$$

We are now ready to present the main theorem for the feasible step sizes for the first-order sampling of two-band signals.

Theorem 2.2 Suppose a signal $f(t)$ is bandpassed over $[d,c]\cup[a,b]$. Let ω_s and T be the sampling frequency and sampling interval, respectively, and $\omega_s=\frac{2\pi}{T}$. Then, $f(t)$ can be completely determined from its samples $f(nT)$ via

$$f(t)=\sum_{n=-\infty}^{\infty} f(nT)\begin{Bmatrix}\frac{2\sigma_1}{\omega_s}\mathrm{csinc}_{[\omega_1,\sigma_1]}(t-NT)+\\ \frac{2\sigma_2}{\omega_s}\mathrm{csinc}_{[\omega_2,\sigma_2]}(t-NT)\end{Bmatrix} \qquad (6)$$

if the sampling frequency ω_s satisfies the following feasibility condition,

$$\omega_s\in\bigcap_{k=1}^{6}\left\{\bigcup_{n=0}^{t_k-1}\left[\frac{b_k-d_k}{t_k+1-n},\frac{a_k-c_k}{t_k-n}\right]\cup\left[b_k-d_k,\infty\right)\right\} \qquad (7)$$

where $[a_k,b_k]$ and $[d_k,c_k]$ is one possible pairing from the group of four intervals corresponding to the two-band case $\{[-b,-a],[-c,-d],[d,c],[a,b]\}$, and

$$t_k=\left\lfloor\frac{a_k-c_k}{(b_k-a_k)+(c_k-d_k)}\right\rfloor,k=1,\ldots,6.$$

Proof: First, we introduce an impulse train modulated by the samples $f(nT)$ of the signal $f(t)$:

$$f_\delta(t)=\sum_{n=-\infty}^{\infty}Tf(nT)\delta(t-nT).$$

According to the Poisson formula, the Fourier transform of $f_\delta(t)$ is given by

$$F_\delta(\omega)=\sum_{n=-\infty}^{\infty}Tf(nT)e^{-jnT\omega}=\sum_{n=-\infty}^{\infty}F(\omega+n\omega_s) \qquad (8)$$

where $F(\omega)$ is the Fourier transform of $f(t)$. The spectrum of $f(t)$ can be recovered from (8) by applying the ideal two-band bandpass filter (4) to $F_\delta(\omega)$ as follows:

$$F(\omega)=S(\omega)F_\delta(\omega)=S(\omega)\sum_{n=-\infty}^{\infty}F(\omega+n\omega_s). \qquad (9)$$

This is guaranteed because none of the spectra $F(\omega+n\omega_s)$, $n\ne 0$, overlap with $F(\omega)$ at the sampling rate ω_s satisfying (7) according to Theorem 2.2. Therefore, taking the inverse Fourier transform of (9) yields

$$f(t)=s(t)*f_\delta(t)=$$

$$\sum_{n=-\infty}^{\infty} f(nT)\begin{Bmatrix}\frac{\sigma_1 T}{\pi}\mathrm{csinc}_{[\omega_1,\sigma_1]}(t-NT)+\\ \frac{\sigma_2 T}{\pi}\mathrm{csinc}_{[\omega_2,\sigma_2]}(t-NT)\end{Bmatrix}$$

$$=\sum_{n=-\infty}^{\infty} f(nT)\begin{Bmatrix}\frac{2\sigma_1}{\omega_s}\mathrm{csinc}_{[\omega_1,\sigma_1]}(t-NT)+\\ \frac{2\sigma_2}{\omega_s}\mathrm{csinc}_{[\omega_2,\sigma_2]}(t-NT)\end{Bmatrix}$$

3 A fast algorithm on intervals

It is observed from (7) that the interval(s) for feasible sampling rates are calculated from the intersections of different groups of intervals. The number of intervals could be large in (7), hence increase the amount of computation significantly. In this section, we present a fast algorithm for computing the intersections among given intervals. This algorithm takes advantage of the facts that the set of all feasible intervals for each pair of bands is a union of disjoint intervals and the feasible intervals are ordered along the number line (frequency axis).

We present the algorithm in the form of a pseudo-code as follows:

Initialization:

Let each of the sets $I_{1,n_1}, I_{2,n_2}, ..., I_{k,n_k}$ be a union of closed disjoint intervals and the intervals in each set are ordered, and I_{i,n_i} is the ith set containing n_i such intervals. Thus,

$I_{1,n_1} = [a_{1,1}, b_{1,1}] \cup [a_{1,2}, b_{1,2}] \cup ...[a_{1,n_1}, b_{1,n_1}]$

$I_{2,n_2} = [a_{2,1}, b_{2,1}] \cup [a_{2,2}, b_{2,2}] \cup ...[a_{2,n_2}, b_{2,n_2}]$

$\vdots$

$I_{k,n_k} = [a_{k,1}, b_{k,1}] \cup [a_{k,2}, b_{k,2}] \cup ...[a_{k,n_k}, b_{k,n_k}]$

Step 1.

:if one of the unions is empty, then stop.

:find the maximum value of the left endpoint of the first interval in each union call it L_{max},

:find the minimum value of the right endpoint of the first interval in each union call it R_{min}.

Step 2.

:If $L_{max} \le R_{min}$, then $[L_{max}, R_{min}]$ is an intersection and is stored in F. Remove the interval that R_{min} occurred in from the union it was contained in, and go to Step 1.

:Else remove the interval that R_{min} occurred in from the union it was contained in and go to Step 1.

Let $C\{I_{1,n_1}, I_{2,n_2}, ..., I_{k,n_k}\}$ represent the number of symbolic operations on the intervals required for the selection process, an upper bound (worst case scenario) on the complexity of this algorithm is given as follows

$$C\{I_{1,n_1}, I_{2,n_2}, ..., I_{k,n_k}\} \le (2k-1)\left[\sum_{i=1}^{k}(n_i - 1) + 1\right] \qquad (10)$$

A program is made in MATLAB to realize this algorithm. In most experiment, the number of operations is much less than the upper bound in (10) because some I_{i,n_i} runs out quickly during the process.

4 Discussion and Conclusion

It is observed from (2) that the bigger the gap between the two passbands the greater the integer t, which implies that there is a greater opportunity for the gap between the passbands to be utilized for larger sampling interval (step size). In other words, the more scattered the passbands, the higher possibility for lower sampling rate.

Consider a bandpass signal with the following band positions $[20kHz, 25kHz] \cup [500kHz, 520kHz]$, then the Nyquist rate is $1040kHz$. However, with the proposed sampling algorithm, the sampling rate is calculated as low as $50kHz$. The difference is significant.

Lower sampling rates are achievable for bandpass signals with significant gaps between the bands. In this paper, the intervals for feasible sampling rates, including the optimal rates, are presented in closed forms. These rates can be proved to be necessary for admissible sampling (anti-aliasing) too. It is not difficult to extend the results to bandpass signals with arbitrary number of bands. However, it is forewarned that the proposed algorithms are effective for bandpass signals with significant gaps in between the bands. Otherwise, imperfections of sampling could lead to aliasing if the margin of error is small, particularly at the optimal sampling rates.

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