Self-Similarity and Long-Range Dependence in Teletraffic

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Abstract: - This paper revisits three important concepts in fractal type network traffic, namely, self-similarity (SS), long-range dependence (LRD), and local self-similarity (LSS). Based on those concepts, we address the reason why the local properties of fractional Gaussian noise (fGn) are contained in the global properties of fGn and vice versa, which may be a limitation of fGn in data traffic modeling.

Key-Words: - Fractal time series; Network traffic; Fractal dimension; The Hurst parameter.

1 Introduction
Let \( \{x(t)\}(-\infty < t < \infty) \) be a random process, where \( x(t) \) is a sample function (\( l = 1, 2, \ldots \)). Without confusions causing, we regard \( x(t) \) as a random function instead of using the symbol \( \{x_l(t)\} \).

Three categories of fractal time series have been used in network traffic (traffic for short). Namely, SS series/process (Tsybakov and Georganas [1], Leland et al. [2], Stalling [76], Carmona et al. [77], Pitts and Schormans [78], and MaDysan [79]), LRD series/process (Beran [3,4], Beran et al. [5], Karagiannis et al. [6], Li [7], and Li et al. [8]), and LSS series/process (Li and Lim [9]). Among three, a class of stationary SS processes called the fractional Gaussian noise (fGn) introduced by Mandelbrot and Ness [10] is widely used, see e.g., [1-6], Paxson and Floyd [11], Willinger et al. [12], Li et al. [13], and Li [14,15], Tsybakov and Georganas [16], Li and Zhou [17], Ada [18], Stoever et al. [19], Lee and Fapojuwo [20], Paxson [21], Jeong et al. [22], Ledesma and Liu [23], Li and Chi [24], Horna et al. [25], Lopez-ardao et al. [26], Park and Tuan [27], Erramilli et al. [28], Michiel and Laevens [29], Willinger and Paxson [30], and Shen et al. [31].

Two reasons why fGn can be widely used in traffic modeling may be as follows. The first reason may be that it is fGn that came to this world earliest with the satisfactorily mathematical theory ([3], [10], Mandelbrot [32]). The second is its simplicity in the expression of its autocorrelation function (ACF) so that two important properties of traffic, SS and LRD, are characterized by a single Hurst parameter (H).

Nevertheless, fGn may not be enough to model traffic due to the complex nature of traffic in multi-scale and multi-fractal, see e.g., [1,9,11,15,17], Abry et al. [33], Feldmann et al. [34], Li et al. [35], Abry and Veitch [36], and Gong et al. [37].

Due to the considerable effects of SS, LRD and LSS on the performances of network systems, see e.g., [14-18], [27], Erramilli et al. [38], Veres et al. [39], Fonseca et al. [40], Park et al. [41], Melo and Fonseca [42], Scherrer et al. [43], Ng et al. [44], Kim et al. [45], Carpio [46], Nain [47], Rolls et al. [48], and Norros [49], revisiting those concepts may be beneficial to applying fractal time series to practice.

In the next section, we shall address the concepts of the processes of SS, LRD, and LSS, which is followed by conclusions.

2 SS, LRD, and LSS
The property of SS of \( x(t) \) describes the self-similar behavior at all time scales while LRD its global behavior at large time scales only. More precisely, SS concerns with the degree that \( x(t) \) is statistically similar with \( x(t + \tau) \) for all lags. On the other side, LRD characterizes the extent that \( x(t) \) correlates to \( x(t + \tau) \) only for large \( \tau \). Thus, they are different concepts. Just as its name implies, it may not make a sense to consider the LRD of \( x(t) \) for small \( \tau \).

In general, a process with LRD may not be SS. The only exception is fGn that is SS and can be LRD (Wolperta and Taqqu [50], Lim and Muniandy [51]). That is, fGn happens to be the only case that reflects its property SS in its global property LRD in the field of stationary fractal time series with LRD and vice versa (Mandelbrot [52, p. 27]). However, in practice, it must be too much for one to expect that there be an exact SS for a general stationary fractal time series with LRD in nature. Thus, the concept of LSS is needed. For any other stationary fractal time series with LRD, they are not SS in the exact sense as fGn is but they may be LSS. Owing to the fact that fGn is so widely used in various fields, the concept of LSS is particularly useful for LRD processes.

2.1 SS Series
Denote two real numbers by \( a > 0 \) and \( 0 < H < 1 \). If \( x(t) \) is such that it has the same statistical distribution with \( a^{-H}x(at) \), we say that \( x(t) \) is an SS series with the SS degree \( H \). This is written by
\[
x(t) = a^{-H}x(at), \quad t > 0.
\]  
Eq. (1) implies SS in the exact sense. More precisely, \( (x(t_1), \ldots, x(t_i)) \) has the same distribution with \( (a^{-H}x(at_1), \ldots, a^{-H}x(at_i)) \) for all \( i > 0 \).

An SS series has the following properties.

P1. Mean: \( \text{E}[x(t)] = a^{-H}\text{E}[x(at)] \).

P2. Variance: \( \text{Var}[x(t)] = \text{Var}[a^{-2H}\text{E}[x(at)]] \).

P3. Autocorrelation function (ACF):
\[
R(t, s) = a^{2H}R(at, as).
\]  
Note that the dynamics of traffic are so complex that its variance may not exist in general (Willinger et al. [53]). Since the statistics of a Gaussian process are completely characterized by its ACF (Papoulis and Pillai [54]), the following discussions are mainly in the domain of Gaussian processes and the main interest is the ACF of traffic.

Denote the standard Brownian motion (BM) by \( B(t) \) for \( t \geq 0 \) and \( B(0) = 0 \). Let \( B_H(t) \) be the fractional Brownian motion (fBm) defined by (Mandelbrot [10,32])
\[
B_H(t) - B_H(0) = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{t} \left[ (t-u)^{H-0.5} - (-u)^{H-0.5} \right] dB(u) + \int_{0}^{t} (t-u)^{H-0.5} dB(u).
\]  
Obviously, \( B_H(t) \) is SS.

According to the properties of BM, one can obtain the following ACF of \( B_H(t) \)
\[
\rho[B_H(t_1), B_H(t_2)] = \frac{\sigma^2}{2} \left[ (t_1)^{2H} + (t_2)^{2H} - (t_2 - t_1)^{2H} \right].
\]  
As \( \rho[B_H(t_1), B_H(t_2)] \) is time varying, \( B_H(t) \) is a nonstationary process with SS.

Let \( G(t) = B_H(t + a) - B_H(t), \ a \in \mathbb{R}, \) be the increment process of \( B_H(t) \). Then, \( G \) is fGn. Clearly, fGn is SS. The ACF of fGn in the continuous case is given by (Mandelbrot [10], Li and Lim[55])
\[
r(\tau) = \sigma^2 \left[ \frac{k}{\epsilon} \right]^{2H} + \frac{k}{\epsilon} - 2 \frac{k}{\epsilon}^{2H},
\]  
where \( \sigma^2 = (1-2H)\cos(H\pi)/H\pi \) is the intensity of fGn, \( H \in (0, 1) \), and \( \epsilon > 0 \) is used by smoothing fBm so that the smoothed fBm is differentiable, see [10, p. 427-428] for details. The ACF of the discrete fGn in the normalized case is expressed by
\[
r(k) = 0.5[(\tau + 1)^{2H} - 2^{2H} + (\tau - 1)^{2H}].
\]  
Since \( r(k) \) does not rely on time, fGn is a stationary process with SS.

The right side of (8) is the finite second-order difference of \( 0.5(\tau)^{2H} \). Approximating it with the second-order differential of \( 0.5(\tau)^{2H} \) yields
\[
r(k) \approx H(2H-1)(\tau)^{2H-2}.
\]  
For \( k \geq 10 \), \( H(2H-1)\tau^{2H-2} \) is a quite satisfactory approximation of \( r(k) \) (Mandelbrot [56]).

Note 1. The above discusses the concept of SS without relating to the concept of LRD.

2.2. LRD Series

Let \( X \) be a stationary process. Its ACF has the following asymptotic property,
\[
r(k) \sim c k^{-2H}/2(k \to \infty), \ H \in (0,1).
\]  
Then, \( r(k) \) is non-summable for \( 0.5 < H < 1 \) and summable for \( 0 < H < 0.5 \). The former corresponds to LRD series while later short-range dependent (SRD) series [3, p. 42].

Replacing \( H(2H-1) \) by \( c \) on the right side of Eq. (9), one immediately sees that fGn is LRD for \( 0.5 < H < 1 \) and SRD for \( 0 < H < 0.5 \). fGn is uncorrelated at any two different time points in the case of \( H = 0.5 \), meaning that fGn becomes the white noise if \( H = 0.5 \).

Note that a stationary random function can be regarded as the output \( y(t) \) of a linear filter that is under the excitation of white noise \( w(t) \). Denote \( h(t) \) the impulse function of a linear system ([24], Robinson [57], Mortensen [58], Parzen [59]). Then,
\[
y(t) = \int_{0}^{t} h(t-\tau)w(\tau)d\tau.
\]  
On the other side, a nonstationary random function can be taken as the output of a time varying linear filter under the excitation of \( w(t) \). Therefore, designing various filters may yield different random functions with the help of \( w(t) \). Hence, conventionally, one considers \( w(t) \) as the headspring or root of random functions (Press et al. [60]).

Since Mandelbrot and van Ness introduced fGn [10], things change. The white noise that plays a primogenitor role in random functions degrades to the son of fGn at once. Such a fact suffices to exhibit the significant advance of fGn in the field of time series. In fact, the applications of fGn with LRD are reported in many fields of sciences and technologies, ranging from network traffic to hydrology, see e.g., [3], [32], [52,53], Bassingthwaighte et al. [61], Korvin [62], Peters [63], Levy-Vehel et al. [64], Liu and Chang [65], Chang et al. [66], Fortin et al. [67], Bickel et al. [68], Herman et al. [69], Jennane et al. [70], Kumar [71], Wel et al. [72], Jin et al [73], Kim et al. [74], Koutsoyiannis [75], and references...
therein. Nevertheless, expecting all LRD series to obey the autocorrelation rule given by Eq. (8) seems unrealistic. Traffic may be the one of the cases that the fGn may not be a model accurate enough [1,7,11,80].

Mathematically, any series whose ACF is non-summable in the discrete case or non-integrable in the continuous case is LRD. Therefore, the class of LRD processes is rich. FGN is only a class of LRD processes. There are other models with LRD. For instance, Li et al. [8] proposes the following ACF that is non-summable to describe LRD series,

$$R(k) = (k + 1) - \beta + Lu(k - m),$$

where \(u(k)\) is the unit step function \((\beta > 0, 0 \leq L < 1, m = 1, 2, ..., k = 0, 1, 2, \ldots)\). Other LRD models can be seen from Li and Lim [9], Martin and Walker [81], Martin and Eccleston [82], Gneiting [83], Ma [84], Granger [85]. Those are LRD but not SS.

**Note 2.** The concept of LRD is not related to that of SS. □

### 2.3 LSS Series

Generally, it may be too much to desire all fractal time series to follow (1) for all time scales \(\tau > 0\) (Mandelbrot [86]). In fact, in Samorodnitsky and Taqqu book [87], they gave an argument to show that it is impossible for any stationary Gaussian processes to be SS. However, they can be LSS.

Kent and Wood [88] gives the definition of LSS as follows. Let \(X(\tau)\) be a stationary Gaussian process and \(C(\tau)\) be its normalized ACF. Then, for a constant \(\beta > 0\), if \(C(\tau)\) for \(\tau \to 0\) satisfies the following

$$C(\tau) = 1 - A|x|^{\alpha}(1 + O(|\tau|^\beta)), \quad 0 < \alpha < 2,$$

\(X(\tau)\) is called the stationary Gaussian process of \(\alpha\) order of LSS. The class of Gaussian processes satisfying (12) is known as Adler processes (Adler [90], Lang and Roueff [91], Lim and Li [92]). Another equivalent definition of LSS is like this. If

$$X(s) - X(r \tau) = r^{-\beta}[X(s) - X(\tau)], \quad |s - r \tau| \to 0,$$

\(X(t)\) is said to be locally self-similar of index \(\nu\). Thus, LSS can be investigated by \(C(\tau)\) for \(\tau \to 0\), i.e., for small \(\tau\). The traffic model of the generalized Cauchy process is LSS (Li and Lim [9]).

**Note 3.** An SS process is LSS whereas the inverse is untrue. □

### 3 Conclusion

We have explained that the concept of SS differs from that of LTD. Except fGn, LRD processes may be LSS. These may be useful to interpret traffic models with LRD in addition to fGn.

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