Multitime Hamilton-Jacobi Theory

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Abstract: This paper combines some of our ideas to obtain the multitime variants of Hamilton-Jacobi theory. Section 1 introduces some Riemannian first order jet bundles. Section 2 describes the multitime Hamilton-Jacobi PDE system connected to the path independent curvilinear integral action. Section 3 analyzes the multitime Hamilton-Jacobi divergence PDE connected to multiple integral action. Section 4 studies the harmonic map Lagrangian. Section 5 underlines the novelty of our results.

Key Words: Hamilton-Jacobi PDE, path independent curvilinear integral action, multiple integral action, submanifolds.

1 Riemannian first order jet bundles
Let \((T, h)\) be a Riemannian manifold with \(m\) dimensions, and \((M, g)\) be a Riemannian manifold with \(n\) dimensions. Then \((J^1(T, M), h + g + h^{-1} \ast g)\) is a Riemannian first order jet bundle with \(m + n + mn\) dimensions and

\[
\left(J^1(T \times M, T), h + g + h^{-1} \ast h + h^{-1} \ast g\right)
\]

is a Riemannian first order jet bundle with \(m + 2m + mn + n^2\) dimensions. Particularly, the Riemannian manifolds \((R^m, \delta_{ij}), (R^n, \delta_{ij})\) determine the Riemannian first order jet bundles \(J^1(R^m, R^n), J^1(R^m \times R^n, R^m)\).

2 Multitime Hamilton-Jacobi PDE system
In mathematics, the Hamilton-Jacobi PDE is a necessary condition describing extremal geometry in generalizations of calculus-of-variations problems. In physics, the Hamilton-Jacobi PDE is equivalent to Newton Law of Motion, Lagrangian Mechanics and Hamiltonian Mechanics. The Hamilton-Jacobi PDE is particularly useful in identifying conserved quantities for mechanical systems, which may be possible even when the mechanical problem itself cannot be solved completely. We improve this point of view adding the multitime version of the Hamilton-Jacobi PDE (see [3]-[11]).

Let \(t = (t^i) \in R^m\) and \(x = (x^i) \in R^n\). Let \(S : R^m \times R^n \to R\) be a \(C^1\) function to whom we attach the constant level sets \(\Sigma_c : S(t, x) = c\). Suppose \(\Sigma_c\) are submanifolds in \(R^{n+m} = R^m \times R^n\), i.e., the normal vector field \(\left(\frac{\partial S}{\partial \delta^i}, \frac{\partial S}{\partial x^i}\right)\) is not zero anywhere.

Let \(\Gamma : (t, x(t))\) be an \(m\)-sheet transversal to the submanifolds \(\Sigma_c\). Then the function \(c(t) = S(t, x(t))\) has nonzero partial derivatives

\[
\frac{\partial c}{\partial \delta^i}(t) = \frac{\partial S}{\partial \delta^i}(t, x(t))
\]

\[
+ \frac{\partial S}{\partial x^i}(t, x(t)) \frac{\partial x^i}{\partial \delta^j}(t) = \Delta_i(t, x(t), x_j(t)) \neq 0.
\]

We accept \(L_\alpha(t, x(t), x_j(t)) = \Delta_i(t, x(t), x_j(t))\), i.e., the Lagrange 1-form \(L_\alpha\) is just the total derivative of the function \(S(t, x(t))\). It follows

\[
p^\alpha_{\gamma} = \frac{\partial L_\alpha}{\partial x^\gamma} = \frac{\partial \Delta_\alpha}{\partial x^\gamma} = \frac{\partial S}{\partial x^\gamma} \delta^\alpha_{\gamma}
\]

or explicitly,

\[
p^\gamma_{\alpha} = 0 \text{ for } \alpha \neq \gamma \text{ and } p^\alpha_{\gamma} = \frac{\partial S}{\partial x^\gamma} \text{ for } \alpha = \gamma.
\]

The Legendre transformation gives the 1-form Hamiltonian

\[
H_\alpha = x^\gamma p^\gamma_{\alpha} - L_\alpha = x^\gamma \frac{\partial S}{\partial x^\gamma} - L_\alpha.
\]

We denote \(p = (p^\alpha_{\gamma})\), with no summation after the index \(\alpha\). In these conditions, the relation

\[
x^\gamma_\alpha(t) = x^\gamma_\alpha(t, x(t), p(t))
\]
reads
\[ x^i_{\gamma}(t) = x^i_{\alpha}(t, x(t), \frac{\partial S}{\partial x^i}(t, x(t))). \]

On the other hand, the relation
\[ L_\alpha(t, x(t), x_\gamma(t)) = \frac{\partial S}{\partial t^\alpha}(t, x(t)) + \frac{\partial S}{\partial x^i}(t, x(t)) \]
rewrites
\[ -\frac{\partial S}{\partial t^\alpha}(t, x(t)) = \frac{\partial S}{\partial t^\alpha}(t, x(t)) \frac{\partial x^i}{\partial a}(t, x(t)) + \frac{\partial S}{\partial x^i}(t, x(t)) - L_\alpha(t, x(t), x_\gamma(t)) \]
or as a multitime Hamilton-Jacobi PDE system
\[ \frac{\partial S}{\partial t^\alpha} + H_\alpha(t, x, \frac{\partial S}{\partial x}) = 0. \]

As a rule, the multi-temporal Hamilton-Jacobi PDE system is accompanied by the initial conditions \( S(0, x) = S_0(x) \) and by complete integrability conditions (closed 1-form, see also [1], [2]). The solution \( S(t, x) = (S^\alpha(t, x)) \) is called the generator vector of canonical multi-momentum. Conversely, let \( S(t, x) \) be a solution of the multi-temporal Hamilton-Jacobi PDE system. Defining
\[ p_\alpha^\gamma(t) = \frac{\partial S}{\partial x^i}(t, x(t)) p_\alpha^i = 0 \text{ for } \alpha \neq \gamma \]
it appears the relation
\[ \int_{t_0}^{t_1} L_\alpha(t, x(t), x_\gamma(t)) dt^\alpha \]
\[ = \int_{t_0}^{t_1} \left( x^i_{\alpha}(t) \frac{\partial S}{\partial x^i}(t, x(t)) - H_\alpha \right) dt^\alpha \]
\[ = \int_{t_0}^{t_1} \frac{\partial S}{\partial t^\alpha} dt^\alpha + \frac{\partial S}{\partial x^i} dx^i = \int_{t} dS, \]
which shows that the action (path independent) curvilinear integral depends only on the boundary values of \( S \).

### 3 Multitime Hamilton-Jacobi divergence PDE

Let \( t = (t^\alpha) \in R^m \) and \( x = (x^i) \in R^n \). Let \( S = (S^\alpha) : R^m \times R^n \to R^m \) be a \( C^1 \) function to whom we attach the constant level sets \( \Sigma_c : S(t, x) = c \) or \( \Sigma_c : S^\alpha(t, x) = c^\alpha, \alpha = 1, ..., m \). Suppose \( \Sigma_c \) are submanifolds in \( R^{m+n} = R^m \times R^n \), i.e., the normal vector fields \( \left( \frac{\partial S^\alpha}{\partial t^\beta}, \frac{\partial S^\alpha}{\partial x^i} \right) \) are not zero anywhere. Let \( \Gamma : (t, x(t)) \) be an \( m \)-sheet transversal to the submanifolds \( \Sigma_c \). Then the vectorial function \( c(t) = S(t, x(t)) \) has nonzero total divergence
\[ \text{Div } c = \frac{\partial c^\alpha}{\partial t^\alpha}(t) = \frac{\partial S^\alpha}{\partial x^i}(t, x(t)) \]
\[ + \frac{\partial S^\alpha}{\partial x^i}(t, x(t)) \frac{\partial x^i}{\partial t^\alpha}(t) = \Delta(t, x(t), x_\gamma(t)) \neq 0. \]

Now, let us introduce the Lagrangian
\[ L(t, x(t), x_\gamma(t)) = \Delta(t, x(t), x_\gamma(t)). \]

It follows the generalized multi-momentum
\[ p = (p^\alpha_i), p^\alpha_i = \frac{\partial L}{\partial x^i} = \frac{\partial \Delta}{\partial x^i} = \frac{\partial S^\alpha}{\partial x^i} \]
and we accept that these relations define a Legendre duality. In these conditions, the relation \( x^i_\gamma(t) = x^i_\gamma(t, x(t)), p(t) \) becomes
\[ x^i_\gamma(t) = x^i_\gamma(t, x(t), \frac{\partial S}{\partial x^i}(t, x(t))). \]

On the other hand, the relation
\[ L(t, x(t), x_\gamma(t)) = \frac{\partial S^\alpha}{\partial t^\alpha}(t, x(t)) + \frac{\partial S^\alpha}{\partial x^i}(t, x(t)) \frac{\partial x^i}{\partial t^\alpha}(t) \]
is equivalent to
\[ -\frac{\partial S^\alpha}{\partial t^\alpha}(t, x(t)) = p^\alpha_i(t) \frac{\partial x^i}{\partial t^\alpha}(t) - L(t, x(t), x_\gamma(t)). \]

Recognizing the Hamiltonian in the right hand member, we obtain the multi-temporal Hamilton-Jacobi divergence PDE
\[ \frac{\partial S^\alpha}{\partial t^\alpha} + H \left( t, x, \frac{\partial S}{\partial x} \right) = 0. \]

As a rule, the multi-temporal Hamilton-Jacobi PDE is accompanied by the initial conditions \( S^\alpha(0, x) = S_0^\alpha(x) \). The solution \( S(t, x) = (S^\alpha(t, x)) \) is called the generator vector of canonical multi-momentum.
Conversely, let $S(t, x) = (S^\alpha(t, x))$ be a solution of the multi-temporal Hamilton-Jacobi PDE. Defining
\[ p^\alpha_i(t) = \frac{\partial S^\alpha}{\partial x^i}(t, x(t)), \]
it appears the relation
\[ \int_\Omega L(t, x(t), x'(t))dt^1...dt^n \]
\[ = \int_\Omega (x^i_x(t) \frac{\partial S^\alpha}{\partial x^i}(t, x(t)) - H(t, x(t)))dt^1...dt^n \]
\[ = \int_\Omega (x^i_x(t) \frac{\partial S^\alpha}{\partial x^i}(t, x(t)) + \frac{\partial S^\alpha}{\partial t}(t, x(t)))dt^1...dt^n \]
\[ = \int_\Omega \frac{\partial c^\alpha}{\partial t}(t)dt^1...dt^n = \int_\partial \delta_{\alpha \beta} c^\alpha(t) n^\beta(t)ds, \]
which shows that the action multiple integral depends only on the boundary values of $c(t)$.

### 4 Harmonic map Lagrangian

The harmonic map Lagrangian
\[ L = \frac{1}{2} h^{\alpha \beta} g_{ij} x^i_{\alpha} x^j_{\beta} \]
produces the kinetic energy
\[ H = x^i_\alpha \frac{\partial L}{\partial x^i_\alpha} - L = \frac{1}{2} h^{\alpha \beta} g_{ij} x^i_{\alpha} x^j_{\beta} = H. \]

From
\[ \frac{\partial L}{\partial x^i_\alpha} = \frac{\partial S^\alpha}{\partial x^i} = h^{\alpha \beta} g_{ij} x^j_{\beta}, \]
it follows
\[ x^j_{\beta} = h_{\alpha \beta} g^{ij} \frac{\partial S^\alpha}{\partial x^i} \]
and consequently
\[ H = \frac{1}{2} h_{\alpha \beta} g^{ij} \frac{\partial S^\alpha}{\partial x^i} \frac{\partial S^\beta}{\partial x^j}. \]

Finally,
\[ \frac{\partial S^\alpha}{\partial t^\alpha} + \frac{1}{2} h_{\alpha \beta} g^{ij} \frac{\partial S^\alpha}{\partial x^i} \frac{\partial S^\beta}{\partial x^j} = 0. \]  

\[ \text{(2)} \]

The function $S^\alpha(t, x) = c^\alpha_i = c^\alpha_i - J^\alpha(x)$ is a solution of the PDE (2) if and only if
\[ c^\alpha_i = -\frac{1}{2} h_{\alpha \beta} g^{ij} \frac{\partial J^\alpha}{\partial x^i} \frac{\partial J^\beta}{\partial x^j}, \]
c^\alpha_i < 0.

Now, let us analyze the constant level sets $S^\alpha(t, x) = c^\alpha$ or $c^\alpha_i - J^\alpha(x) = c^\alpha$. Suppose they are submanifolds in $R^{n+n}$, i.e., the normal vector fields $N^\alpha_\beta = c^\alpha_\beta - J^\alpha = \frac{\partial J^\alpha}{\partial x^i}$ are nonzero anywhere. For each $t = (t^\beta)$, we have a submanifold of $R^n$ of normal vector fields $N^\alpha_i = \frac{\partial J^\alpha}{\partial x^i}$ or $N^\alpha_\beta = g^{ij} h_{\alpha \beta} N^\beta_j$ of constant norm
\[ ||N||^2 = h^{\alpha \beta} g_{ij} N^\alpha_i N^\beta_j = h_{\alpha \beta} g^{ij} \frac{\partial J^\alpha}{\partial x^i} \frac{\partial J^\beta}{\partial x^j} = -2 c^\alpha > 0. \]

Coming back to the general situation, let $S(t, x) = (S^\alpha(t, x))$ be a general solution of the multi-temporal Hamilton-Jacobi PDE (1), where
\[ H(t, x, p) = x^i_\alpha p^\alpha_i - L(t, x, x'), \]
\[ p^\alpha_i = \frac{\partial L}{\partial x^i_\alpha}. \]

Taking the partial derivative with respect to $x^j_\beta$, we find
\[ \frac{\partial H}{\partial p^\alpha_i} = p^\beta_\alpha + x^i_\alpha \frac{\partial p^\alpha_i}{\partial x^i_\beta} - \frac{\partial L}{\partial x^i_\beta}; \]
\[ \frac{\partial H}{\partial \frac{\partial^2 L}{\partial x^i_\alpha \partial x^i_\beta}} = p^\beta_\alpha + x^i_\alpha \frac{\partial^2 L}{\partial x^i_\alpha \partial x^i_\beta} = -x^i_\alpha \frac{\partial^2 L}{\partial x^i_\alpha \partial x^i_\beta}. \]

We denote $G^{-1}_{ij} = -\frac{\partial^2 L}{\partial x^i_\alpha \partial x^i_\beta}$. Supposing
\[ \text{det } G^{-1}_{ij} \neq 0, \]
we follow
\[ x^i_\alpha = \frac{\partial H}{\partial p^\alpha_i}. \]  

\[ \text{(3)} \]

Let us consider a solution of the PDE system
\[ \frac{\partial x^i_\alpha}{\partial t^\alpha} = \frac{\partial H}{\partial p^\alpha_i}, \]
\[ p^\alpha_i = \frac{\partial S^\alpha}{\partial x^i_\alpha}, \]  

\[ \text{(4)} \]

where $S^\alpha$ are functions of class $C^2$. We compute the total divergence
\[ \frac{\partial p^\alpha_i}{\partial t^\alpha} = \frac{\partial^2 S^\alpha}{\partial x^i_\alpha \partial x^i_\alpha} p^\alpha_i + \frac{\partial^2 S^\alpha}{\partial x^i_\alpha \partial x^i_\alpha} \frac{\partial H}{\partial p^\alpha_i}, \]
\[ \frac{\partial^2 S^\alpha}{\partial x^i_\alpha \partial x^i_\alpha} \frac{\partial H}{\partial p^\alpha_i} + \frac{\partial^2 S^\alpha}{\partial x^i_\alpha \partial x^i_\alpha} \frac{\partial H}{\partial p^\alpha_i} = 0. \]

On the other hand, taking the partial derivative of (1) with respect to $x^j_\beta$, we find
\[ \frac{\partial^2 S^\alpha}{\partial x^i_\alpha \partial x^i_\alpha} + \frac{\partial H}{\partial p^\alpha_i} \frac{\partial H}{\partial p^\alpha_i} \frac{\partial^2 S^\alpha}{\partial x^i_\alpha \partial x^i_\alpha} = 0. \]
From the last two relations we obtain
\[ \frac{\partial p_i^\alpha}{\partial t} = -\frac{\partial H}{\partial x^i}. \]

Consequently the solutions of the PDE system (4) are solutions of the Hamilton PDE
\[ \frac{\partial x^i}{\partial t} = \frac{\partial H}{\partial p_i^\alpha}, \quad \frac{\partial p_i^\alpha}{\partial t} = -\frac{\partial H}{\partial x^i}. \]

On the other hand, the relation (3) shows that the PDE system (4) is equivalent to
\[ x^i = \frac{\partial L}{\partial x^i}, \quad p_i^\alpha = \frac{\partial L}{\partial x^i} = \frac{\partial S^\alpha}{\partial x^i}. \]

In the case
\[ L = \frac{1}{2} h^{\alpha\beta} g_{ij} x^i x^j - V(t, x), \]

it follows
\[ p_i^\alpha = \frac{\partial L}{\partial x^i} = h^{\alpha\beta} g_{ij} x^j = \frac{\partial S^\alpha}{\partial x^i}, \]

and hence
\[ \frac{\partial x^i}{\partial t} = h_{\alpha\beta} g_{ij} \frac{\partial S^\beta}{\partial x^j}. \]

In this way, the PDE system (4) describes the submanifolds which are orthogonal to the submanifolds
\[ S^\alpha(t, x) = c^\alpha, \]

for each \( t \). The orthogonality is given by the Riemannian structure
\[ T(t, x, x_\gamma) = h^{\alpha\beta}(t) g_{ij}(t, x)x^i x^j, \]

with the parameter \( t \).

**Theorem.** Given a mechanical system with the Lagrangian (5) and a solution \( S^\alpha \) of the multi-temporal Hamilton-Jacobi PDE (1) associated to the mechanical system, the submanifolds orthogonal to the submanifolds \( S^\alpha(t, x) = c^\alpha \), in the sense of the Riemannian structure
\[ C_{ij}^{\alpha\beta}(t, x) = h^{\alpha\beta}(t) g_{ij}(t, x), \]

are \( m \)-sheets of evolution of the mechanical system.

## 5 Open problem

Develop a theory similar to those of Section 4 for the least squares Lagrangian (see geometric dynamics [3]-[11])
\[ L = \frac{1}{2} h^{\alpha\beta} g_{ij}(x^i_\alpha - X^i_\alpha)(x^j_\beta - X^j_\beta). \]

## 6 Conclusions

The multitime Hamilton-Jacobi PDE system is an over-determined system, namely we have more equations that the dimension of solution’s range. Hence, it is reasonable to look for sufficient conditions on the data in order that a solution does exist (see, complete integrability conditions; closed 1-forms).

The multitime Hamilton-Jacobi divergence PDE is an under-determined PDE, namely we have one equation with \( m \) unknown functions. Hence, a solution does always exist.

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