Abstract: - We constructed a new fifth order five-stage singly diagonally implicit Runge-Kutta (DIRK) method which is specially designed for the integrations of linear ordinary differential equations (LODEs). The restriction to linear ordinary differential equations (ODEs) reduces the number of conditions which the coefficients of the Runge-Kutta method must satisfy. Having achieved a particular order of accuracy, the best strategy for practical purposes would be to choose the coefficients of the Runge-Kutta methods such that the error norm is minimized. Thus, here the error norm is minimized so that the free parameters chosen are obtained from the minimized error norm. A set of test problems are used to validate the method and numerical results show that the new method is more efficient in terms of accuracy compared to the existing method.

Key-Words: - Runge-Kutta, Linear ordinary differential equations, Error norm.

1 Introduction
We consider the numerical integration of linear inhomogeneous systems of ordinary differential equations (ODEs) of the form

\[ y' = Ay + G(x) \]  

(1)

where \( A \) is a square matrix whose entries do not depend on \( y \) or \( x \), and \( y \) and \( G(x) \) are vectors. Such systems arise in the numerical solution of partial differential equations (PDEs) governing wave and heat phenomena after application of a spatial discretization such as finite-difference method.

Explicit Runge-Kutta method is very popular for simulations of wave equations; see Zingg and Chisholm [6], due to their high accuracy and low memory requirements.

To derive Runge-Kutta (RK) methods, we need to fulfill certain order equations; see Dormand [3]. These order equations resulted from the derivatives of the function \( y' = f(x, y) \) itself.

If the function is linear then some of the error equations resulted by the nonlinearity in the derivative function can be removed, thus less order equations need to be satisfied, hence a more efficient method in some respect than the classical method can be derived.

In this paper, we construct diagonally implicit Runge-Kutta method specifically for linear ODEs with constant coefficients. We consider the principal terms of the local truncation error to minimize the error norm. Then, a few test equations are used to validate the new method.

2 Materials and Methods
2.1 Derivation of the method
In this section, we consider the following scalar ODE

\[ y' = f(x, y) \]  

(2)

When a general \( s \)-stage diagonally implicit Runge-Kutta method is applied to the ODE, the
following equations are obtained,
\[ y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i \] (3)
where
\[ k_i = f(x_n + c_i h, y_n + h \sum_{j=1}^{i} a_{ij} k_j) \] (4)

We shall always assume that the row-sum condition holds where.

According to Dormand [3], there are 17 order equations (error equations) needed to be satisfied by the fifth order five-stage RK method. The restriction to linear ODEs reduces the number of equations which the coefficients of the RK method must satisfy see Zingg and Chisholm [6]. The order equations are eliminated by exploiting the fact that, for linear ODEs,
\[ \sum_{i} = 0 \]

Zingg and Chisholm [6] too have derived a new explicit RK methods which are suitable for linear ODEs that are more efficient than the conventional RK methods.

Table 1: Order equations for fifth order Runge-Kutta method suitable for LODEs

| 1. | \[ \tau_1^{(1)} = \sum_{i} b_i - 1 \] |
| 2. | \[ \tau_1^{(2)} = \sum_{i} b_i c_i - \frac{1}{2} \] |
| 3. | \[ \tau_1^{(3)} = \sum_{i} b_i c_i^2 - \frac{1}{2} \] |
| 4. | \[ \tau_2^{(3)} = \sum_{j} b_{ij} a_j c_j - \frac{1}{6} \] |
| 5. | \[ \tau_1^{(4)} = \sum_{i} b_i c_i^3 - \frac{1}{4} \] |
| 6. | \[ \tau_3^{(4)} = \sum_{j} b_{ij} a_j c_j^2 - \frac{1}{12} \] |
| 7. | \[ \tau_4^{(4)} = \sum_{jk} b_{ij} a_j a_k c_k - \frac{1}{24} \] |

8. \[ \tau_1^{(5)} = \frac{1}{24} \sum_{i} b_i c_i^4 - \frac{1}{120} \]
9. \[ \tau_5^{(5)} = \frac{1}{6} \sum_{i} b_i a_{ij} c_j^3 - \frac{1}{120} \]
10. \[ \tau_8^{(5)} = \frac{1}{2} \sum_{i} b_i a_{ij} a_{jk} c_k^2 - \frac{1}{120} \]
11. \[ \tau_9^{(5)} = \sum_{i} b_i a_{ij} a_{jk} a_{km} c_m - \frac{1}{120} \]

Using the simplifying assumption:
\[ \sum_{i} b_i a_{ij} c_j = b_j (1 - c_j), \quad j = 2, \ldots, 5 \] (5)

We have
\[ \sum_{i} b_i a_{ij} c_j = \frac{1}{6} \rightarrow \left( \sum_{i} b_i c_i = \frac{1}{2} \right) - \left( \sum_{i} b_i c_i^2 = \frac{1}{2} \right), \]

thus equation 4 can be removed, similarly we can remove equations 6 and 9 in table 1. Thus, using (5) the order equations are replaced by simpler equations. They are:

\[ j = 2 \rightarrow b_3 a_{32} + b_4 a_{42} + b_5 a_{52} = b_2 (1 - c_2 - \gamma) \]
\[ j = 3 \rightarrow b_4 a_{43} + b_5 a_{53} = b_3 (1 - c_3 - \gamma) \]
\[ j = 4 \rightarrow b_5 a_{54} = b_4 (1 - c_4 - \gamma) \]
\[ j = 5 \rightarrow c_5 = 1 - \gamma \]

Altogether there are 11 equations needed to be satisfied and we have 15 unknowns. So, we can have four free parameters which are chosen to be \( c_2, c_3, c_4 \) and \( \gamma \). Solving which, we have all equations in terms of \( c_2, c_3, c_4 \) and \( \gamma \).

The order equations for the sixth order method are the 11 order equations in table 1 and the additional order equations given in table 2 as obtained by Zingg and Chisholm [6].
Table 2: Additional order equations for sixth order Runge-Kutta method

<table>
<thead>
<tr>
<th>i</th>
<th>( \tau^{(6)}_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>( \tau^{(6)}<em>{12} = \sum</em>{i} b_i c_i^2 - \frac{1}{6} )</td>
</tr>
<tr>
<td>13</td>
<td>( \tau^{(6)}<em>{13} = \sum</em>{ij} b_{ij} a_{ij} c_j^4 - \frac{1}{30} )</td>
</tr>
<tr>
<td>14</td>
<td>( \tau^{(6)}<em>{14} = \sum</em>{ijk} b_{ijk} a_{ijk} c_k^6 - \frac{1}{120} )</td>
</tr>
<tr>
<td>15</td>
<td>( \tau^{(6)}<em>{15} = \sum</em>{ijk} b_{ijk} a_{ijk} c_k^6 - \frac{1}{360} )</td>
</tr>
<tr>
<td>16</td>
<td>( \tau^{(6)}<em>{16} = \sum</em>{ijkmn} b_{ijkmn} a_{ijkmn} c_m c_n - \frac{1}{720} )</td>
</tr>
</tbody>
</table>

In order to choose the free parameters \( c_2, c_3, c_4 \) and \( \gamma \), the principal terms of the local truncation error must be considered. Using the error function \( \varphi_p = \sum_{j=1}^{n_p} \tau^{(p+1)}_j F^{(p+1)}_j \) and RK error coefficients [3], the principal term for fifth order method is

\[
\varphi_5 = \sum_{j=1}^{6} \tau^{(6)}_j F^{(6)}_j
\]

The best strategy for practical purposes would be to choose the free RK parameters is to minimize the error norm, see [3];

\[
A^{(p+1)} = \| e^{(p+1)} \|_2 = \sqrt{\sum_{j=1}^{n_p} (r_j^{(p+1)})^2}
\]

So we have the principal error norm for this method;

\[
A^{(6)} = \| e^{(6)} \|_2 = \sqrt{(\tau^{(6)}_1)^2 + (\tau^{(6)}_7)^2 + (\tau^{(6)}_{15})^2 + (\tau^{(6)}_{19})^2 + (\tau^{(6)}_{20})^2}
\]

where \( \tau^{(6)}_j \) are the error equations associated with the sixth order method, (in table 2). Substituting the free parameters into \( A^{(6)} \), we obtained the principal error norm in terms of \( c_2, c_3, c_4 \) and \( \gamma \).

Minimizing the error norm, we have

\[
\begin{align*}
c_2 &= 0.2850628601 833133, \\
c_3 &= 0.4813089538 861712,
\end{align*}
\]

\[
\gamma = 0.070125720366624
\]

\[
\begin{align*}
c_2 &= 0.2850628601 018331 \\
c_3 &= 0.4813089538 861667
\end{align*}
\]

\[
\begin{align*}
c_4 &= 0.7048320137 465169 \quad \text{and} \\
c_5 &= 0.29987427963338
\end{align*}
\]

\[
\begin{align*}
a_{21} &= 0.21493713 981669 \\
a_{31} &= 0.14706690 123068
\end{align*}
\]

\[
\begin{align*}
a_{32} &= 0.26411633228886 \\
a_{41} &= 0.16565616 299779 \\
a_{42} &= 0.18423756 277865
\end{align*}
\]

\[
\begin{align*}
a_{51} &= 0.28481256760346 \\
a_{52} &= 0.26544595147372 \\
a_{53} &= 0.097698357633858
\end{align*}
\]

\[
\begin{align*}
a_{54} &= 0.32625715343955 \\
b_1 &= 0.16938785 743399 \\
b_2 &= 0.21992349 463392 \\
b_3 &= 0.19374055 051289
\end{align*}
\]

\[
\begin{align*}
b_4 &= 0.24674849 025278 \\
b_5 &= 0.17019960 716642
\end{align*}
\]

\[
\begin{align*}
c_1 &= a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = \gamma
\end{align*}
\]

2.2 Stability

One of the practical criteria for a good method to be useful is that it must have region of absolute stability. When an \( s \)-stage Runge-Kutta method (equations (3) and (4)) is applied to the test equation,

\[
y' = f(x, y) = \lambda y
\]

where \( A \) is \( (m \times m) \), \( e \) is \( (m \times 1) \) are obtained from the method itself and \( R(\hat{h}) \) is called the stability polynomial of the method. The stability region is obtained by taking

\[
R(\hat{h}) = R(\hat{h}) = 1 + \hat{h} b^T (I - \hat{h} A)^{-1} e
\]

where \( A \) is \( (m \times m) \), \( e \) is \( (m \times 1) \) are obtained from the method itself and \( R(\hat{h}) \) is called the stability polynomial of the method. The stability region is obtained by taking

\[
R(\hat{h}) = 1 = \cos \theta + i \sin \theta
\]

The stability polynomial is solve for \( \hat{h} \) which
gives the value of \( |\Re(\hat{h})| \leq 1 \); this is done by using Mathematica package (see Torrence [5]). The stability region is obtained by tracing the values of \( \hat{h} \) and is shown in Figure 1. Where the vertical axis is the imaginary part and the horizontal axis is the real part.

![Stability Region](image)

Figure 1: The stability region for the 5th order 5-stage SDIRK method

### 3 Results and Discussion

We use the method to obtain the numerical solutions to the following problems, all of them are linear ODEs.

**PROBLEM 1:**

\[
y'(t) = -y \tan t - \frac{1}{\cos t} \\
y(t) = \cos t - \sin t \\
0 \leq t \leq 1, y(0) = 1
\]

Source: J. C. Butcher [2]

**PROBLEM 2:**

\[
y'(t) = \frac{2}{t} y + t^2 e^t \\
y(t) = t^2 (e^t - e) \\
1 \leq t \leq 5, y(1) = 0
\]

Source: Richard L. Burden and J. Douglas Faires [1].

The numerical results are tabulated and compared with the existing method and below are the notations used:

- **H**: Step size used
- **MTHD**: Method employed
- **MAXE**: Maximum error

\[ |y(x_i) - y_i| \]

- **SDIRK4,5**: Optimal fourth order five-stage SDIRK (Ferracina and Spijker, [4])
- **ERK5L**: Fifth order five-stage explicit RK method for LODEs (Zingg and Chisholm, [6])
- **SDIRK5L**: New fifth order five-stage SDIRK method with minimized error norm for LODEs.

<table>
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<tr>
<th>MTHD</th>
<th>H</th>
<th>MAXE</th>
</tr>
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<tbody>
<tr>
<td>SDIRK4,5</td>
<td>4.90905e-008</td>
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</tr>
<tr>
<td>ERK5L</td>
<td>4.73152e-007</td>
<td>1.72865e-008</td>
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<td>SDIRK5L</td>
<td>6.12153e-009</td>
<td>2.04009e-009</td>
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<td>6.28609e-008</td>
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<tr>
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<td>1.24309e-010</td>
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<td>SDIRK5L</td>
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<td>SDIRK5L</td>
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<td>5.51288e-012</td>
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<tr>
<td>SDIRK5L</td>
<td>3.97293e-012</td>
<td>3.97293e-012</td>
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<td>ERK5L</td>
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<tr>
<td>SDIRK5L</td>
<td>4.44089e-016</td>
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</table>
Table 4: Numerical results for problem 2

<table>
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<tr>
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<tbody>
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<tr>
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<td>4.</td>
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<td>7.</td>
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</table>

4 Conclusion

The new fifth order five-stage SDIRK method with minimized error norm has been presented for the integration of linear ODEs. It has a substantial region of stability, thus, it is stable. From the numerical results in table 3 and 4, we can conclude that the new fifth order five-stage SDIRK method which is suitable for linear ODEs performs better in terms of maximum error compared to the fifth order five-stage ERK method and the optimal fourth order five-stage SDIRK method.

References


