Geodesic and Balanced Bipancyclicity of Hypercubes

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Abstract: For any two vertices \( u, v \in V(G) \), a cycle \( C \) in \( G \) is called a geodesic cycle between \( u \) and \( v \) if a shortest path of \( G \) joining \( u \) and \( v \) lies on the cycle. Let \( G \) be a bipartite graph. For any two vertices \( u \) and \( v \) in \( G \), a cycle \( C \) is called a balanced cycle between \( u \) and \( v \) if \( d_C(u, v) = \max\{d_C(x, y) \mid x \text{ and } y \text{ are in the same partite set, and } y \text{ and } v \text{ are in the same partite set} \} \). A bipartite graph \( G \) is geodesic bipancyclic (respectively, balanced bipancyclic) if for each pair of vertices \( u, v \in V(G) \), it contains a geodesic cycle (respectively, balanced cycle) of every even length of \( k \) satisfying \( \max\{2d_G(u, v), 4\} \leq k \leq |V(G)| \) between \( u \) and \( v \). In this paper, we prove that \( Q_n \) is geodesic bipancyclic and balanced bipancyclic if \( n \geq 2 \).

Key–Words: Hypercube; Interconnection networks; Edge-bipancyclic; Geodesic bipancyclic; Balanced bipancyclic

1 Introduction

An interconnection network topology is usually represented by a graph where vertices represent processors and edges represent links between processors. There are various kinds of graphs applied to design interconnection networks. For example, a ring structure is often used as a connection structure of local area network and as a control and data flow structure in distributed networks due to its good properties such as low connectivity, simplicity, and their feasible implementation [6]. There are a lot of mutually conflicting requirements in designing the topology of computer networks. It is almost impossible to design a network which is optimum in all aspects. Existence of various cycles (rings) in an interconnection network is essential for parallel algorithms that communicate data in token-ring mode. Probably the most effective measure of a communication network performance is the transmission delay encountered by a message in traveling through the network from its source to its destination. In a store-forward network a message may have to be stored and forwarded by several intermediate processors before reaching its destination. The transmission delay is approximately proportional to the number of edges a message must travel.

The \( n \)-dimensional hypercube network, \( Q_n \), is widely used in interconnection network topology based on its many attractive properties such as regularity, recursive structure, node and edge symmetry, maximum connectivity, and effective routing and broadcasting algorithms [6]. Many variations of hypercube network are also proposed, for example, crossed cube, twisted cube, möbius cube, augmented cube, and folded hypercube. Cycles (rings) are one of the most fundamental networks for parallel and distributed computation. They are suitable for designing simple algorithms with low communication costs. Many efficient algorithms designed on rings for solving various algebraic problems and graph problems can be found in [6]. These applications motivate the embedding of various length of cycles in networks. The ring embedding problem, which deals with all possible lengths of cycles in a given graph, is investigated for \( n \)-dimensional hypercube \( Q_n \) [5, 7, 8]. Saad and Schultz [8] proved that \( Q_n \) is bipancyclic in the sense that an even cycle of length \( k \) exists for each even integer between 4 and \( |V(Q_n)| \). Latifi et al [5] found that \( Q_n \) is hamiltonian with up to \( n - 2 \) edge faults. Li et al. [7] proved that \( Q_n \) is still edge-bipancyclic in the sense that every edge of \( Q_n \) lies on a cycle of every even length from 4 to \( |V(Q_n)| \) even if it is up to \( n - 2 \) edge faults.

In this paper, we address the existence of cycles with some properties in \( Q_n \). The rest of this paper is organized as follows. In the next section, we propose notions of geodesic bipancyclic and balanced bipancyclic that are restrictions of the concept of bipancyclic as new measures of cycle embedding capability of a bipartite graph. Section 3 shows that \( Q_n \) is geodesic bipancyclic. Section 4 proves that \( Q_n \) is bal-
2 Preliminaries

Our fundamental graph terminologies refer to [1]. A graph $G = (V, E)$ is bipartite if the node set $V(G) = B \cup W$ is the union of two disjoint node sets $B$ and $W$ (also called the partite sets), such that every edge joins $B$ and $W$. Two vertices, $u$ and $v$, have the same color if and only if $u$ and $v$ are in the same partite set. We also use $G = (B \cup W; E)$ to denote a bipartite graph. Two vertices $a$ and $b$ are adjacent if $(a, b) \in E$. A path is a sequence of adjacent vertices, written as $\langle v_0, v_1, v_2, \ldots, v_m \rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except possibly $v_0 = v_m$. We also write the path $\langle v_0, P[v_0, v_m], v_m \rangle$, where $P[v_0, v_m] = \langle v_0, v_1, \ldots, v_m \rangle$ as well as $v_0$ and $v_m$ are two end-vertices of $P[v_0, v_m]$. The length of a path $P$ denoted by $l(P)$ is the number of edges in $P$. Two paths are vertex-disjoint (also called disjoint) if and only if they do not have any vertices in common. Two edges $(u, v)$ and $(w, z)$ are disjoint if $u \notin \{w, z\}$ and $v \notin \{w, z\}$. Let $u$ and $v$ be two vertices of $G$. The distance between $u$ and $v$ denoted by $d_G(u, v)$ is the length of a shortest path of $G$ joining $u$ and $v$.

A cycle $C$ is a special path with at least three vertices such that the first vertex is the same as the last one. A cycle $C$ is called a $k$-cycle if $l(C) = k$. A path (respectively, cycle) which traverses each vertex of $G$ exactly once is a hamiltonian path (respectively, hamiltonian cycle). They are defined as follows.

**Definition 1** Let $G$ be a graph. For any two vertices $u, v \in V(G)$, a cycle $C$ in $G$ is called a geodesic cycle between $u$ and $v$ if the shortest path joining $u$ and $v$ in $C$ is also a shortest path joining $u$ and $v$ in $G$.

In Definition 1, we define a geodesic $k$-cycle between two distinct vertices, $u$ and $v$, such that the distance of $u$ and $v$ in the cycle is the smallest over all $k$-cycles passing through $u$ and $v$ in $G$. The transmission delay between $u$ and $v$ in this cycle will be the minimum. The geodesic pancyclic property has been first studied by Chan et. al [2]. Recently, the geodesic pancyclicity of Crossed cubes has been proposed by Lai et. al [4].

To route a packet from $u$ to $v$ in a $k$-cycle, one may first breaks the packet into two smaller pieces. Then, route the two pieces along two internal vertex-disjoint paths to the two intermediate vertices $v_1, v_2$. In the second phase, symmetrically, the two pieces are routed from the intermediate vertices $v_1, v_2$ to their common destination $v$. The packet is combined in $v$ until all pieces of this packet arrived. Therefore, this kind of transmission delay between $u$ and $v$ in a cycle is determined by the longest path between $u$ and $v$ in this cycle. It is of interest to find a cycle passing through $u$ and $v$ such that lengths of two disjoint paths between $u$ and $v$ in this cycle are as equal as possible.

**Definition 2** Let $G = (B \cup W; E)$ be a bipartite graph. For any two vertices $u$ and $v$ in $G$, a cycle $C$ is called a balanced cycle between $u$ and $v$ if $d_G(u, v) = \max\{d_G(x, y) \mid x \text{ and } u \text{ are in the same partite set, and } y \text{ and } v \text{ are in the same partite set.}\}$.

A bipartite graph is vertex-bipancyclic (resp., edge-bipancyclic) if every vertex (resp. edge) lies on a cycle of every even length from 4 to $|V(G)|$ inclusive. A bipartite graph $G$ is geodesic bipancyclic (respectively, balanced bipancyclic) if for each pair of vertices $u, v \in V(G)$, it contains a geodesic cycle (respectively, balanced cycle) of every even length of $k$ satisfying $\max\{2d_G(u, v), 4\} \leq k \leq |V(G)|$ between $u$ and $v$. It is observed that every geodesic bipancyclic graph is edge-bipancyclic.

Let $u = u_0 u_1 u_2 \ldots u_{n-1} u_n = 0$ be a n-bit binary strings. The Hamming weight of $u$, denoted by $w(u)$, is the number of $u_i$ such that $u_i = 1$. Let $u = u_{n-1} u_{n-2} \ldots u_1 u_0$ and $v = v_{n-1} v_{n-2} \ldots v_1 v_0$ be two distinct n-bit binary strings. The Hamming distance $h(u, v)$ between two vertices $u$ and $v$ is the number of different bits in the corresponding strings of both vertices. The n-dimensional hypercube, denoted by $Q_n$, consists of all n-bit binary strings as its vertices and two vertices $u$ and $v$ are adjacent if and only if $h(u, v) = 1$. For $0 \leq k < n$, we use $u^k$ to denote the binary string $u$ derived from the binary string $v_{n-1} v_{n-2} \ldots v_1 v_0$ such that $u_k = 1 - v_k$ and $u_i = v_i$ if $i \neq k$. An edge $(u, v) \in E(Q_n)$ is of dimension $i$ if $u = v^i$. It is known that $d_{Q_n}(u, v) = h(u, v)$.

For a given $0 \leq i < n$, we can partition $Q_n$ along dimension $i$ into two $(n-1)$-cubes such that $Q_{n-1}^0$ denotes the subgraph of $Q_n$ induced by $\{x \in V(Q_n) \mid x_i = 0\}$ and $Q_{n-1}^1$ denotes the subgraph of $Q_n$ induced by $\{x \in V(Q_n) \mid x_i = 1\}$. We have $Q_{n-1}^0$ and $Q_{n-1}^1$ being isomorphic to $Q_{n-1}$.

**Lemma 1** [7] Let $u$ and $v$ be two arbitrary distinct vertices with the same partite set in $Q_n$ for $n \geq 2$. Then, for any vertex $w$ such that $h(w, u)$ is odd, there exists a path joining $u$ and $v$ passing all vertices of $Q_n$ except $w$.

**Lemma 2** [7] Let $u$ and $v$ be two arbitrary distinct vertices in $Q_n$ and $h(u, v) = d$, where $n \geq 2$. There are paths formed by $\langle u, P[w, v], v \rangle$ in the $Q_n$ with lengths $d + 2d + 4, \ldots, c$, where $c = 2^n - 1$ if $d$ is odd, and $c = 2^n - 2$ if $d$ is even.
3 Geodesic bipancyclicity of Hypercubes

For any pair of vertices \( u \) and \( v \), a cycle \( C \) is called geodesic cycle between \( u \) and \( v \) in \( Q_n \) if \( h(u,v) = d_{C}(u,v) \). \( Q_n \) is geodesic bipancyclic if \( Q_n \) contains a geodesic cycle of every even length of \( k \) satisfying \( \max\{2h(u,v), 4\} \leq k \leq 2^n \). The following lemma about shortest path properties of \( Q_n \) will be used in the proof of Theorem 1. The proof of Lemma 3 is straightforward. Therefore, it is omitted.

Lemma 3 Let \( u, v \in Q_n \).

(1) If \( u, v \in Q_{n-1}^{0} \) (respectively, \( Q_{n-1}^{1} \)), then there exists a shortest path joining \( u \) and \( v \) in \( Q_n \) with all its vertices in \( Q_{n-1}^{0} \) (respectively, \( Q_{n-1}^{1} \)).

(2) Let \( u \in Q_{n-1}^{0} \) and \( v \in Q_{n-1}^{1} \). Then, (i) There exists a shortest path \( S \) joining \( u \) and \( v \) in \( Q_n \) with all its vertices (except \( u \)) in \( Q_{n-1}^{1} \). (ii) There exists a shortest path \( S \) joining \( u \) and \( v \) in \( Q_n \) with all its vertices (except \( v \)) in \( Q_{n-1}^{0} \).

Theorem 1 \( Q_n \) is geodesic bipancyclic if \( n \geq 2 \).

Proof. Let \( u = u_{n-1}u_{n-2}\ldots u_1u_0 \) and \( v = v_{n-1}v_{n-2}\ldots v_1v_0 \) be any two distinct vertices of \( Q_n \) and \( h(u,v) = d \).

Case 1: \( h(u,v) = 1 \). i.e. \( u \) and \( v \) are adjacent. Obviously, we have that every edge in \( Q_n \) lies on a cycle of every even length from 4 to \( 2^n \). The theorem holds for \( h(u,v) = 1 \).

Case 2: \( h(u,v) = d \geq 2 \). i.e. \( u \) and \( v \) are not adjacent. We prove this theorem by induction on \( n \). Obviously, the theorem holds for \( n = 2 \). Assume that the theorem is true for every integer \( 2 \leq k < n \). To prove this theorem, we establish every geodesic cycle of even length \( k \) between \( u \) and \( v \) in \( Q_n \) where \( 2d \leq k \leq 2^n \). Partitioning \( Q_n \) along dimension 0, we obtain two disjoint \((n-1)\)-subcubes \( Q_{n-1}^{0} \) and \( Q_{n-1}^{1} \) such that \( Q_{n-1}^{0} \) denotes the subgraph of \( Q_n \) induced by \( \{x \in V(Q_n) \mid x_0 = 0\} \) and \( Q_{n-1}^{1} \) denotes the subgraph of \( Q_n \) induced by \( \{x \in V(Q_n) \mid x_0 = 1\} \). The proof of case 2 is divided into two cases: \( u \) and \( v \) are in the same subcube \( Q_{n-1}^{0} \) (or \( Q_{n-1}^{1} \)), and \( u \) and \( v \) are in different subcubes.

Subcase 2-1: \( u, v \in Q_{n-1}^{0} \) or \( u, v \in Q_{n-1}^{1} \).

Without loss of generality, we may assume that \( u \) and \( v \) are in \( Q_{n-1}^{0} \), i.e. \( u_0 = v_0 = 0 \). By induction hypothesis, \( Q_{n-1}^{0} \) is geodesic bipancyclic. \( Q_{n-1}^{0} \) contains every geodesic cycle of even length \( k \) satisfying \( 2d \leq k \leq 2^{n-1} \). Applying Lemma 3, we have that \( d_{Q_{n-1}^{0}}(u,v) = h(u,v) = d \). It is observed that every geodesic cycle between \( u \) and \( v \) in \( Q_{n-1}^{0} \) is a geodesic cycle between \( u \) and \( v \) in \( Q_n \). Thus, a geodesic cycle between \( u \) and \( v \) with even length of \( k \) satisfying \( 2d \leq k \leq 2^{n-1} \) in \( Q_n \) can be found in \( Q_{n-1}^{0} \).

The rest of the proof of this subcase is to find every geodesic cycle of even length from \( 2^{n-1} + 2 \) to \( 2^n \) between \( u \) and \( v \) in \( Q_n \). Let \( C \) be a geodesic cycle of length \( 2^{n-1} \) between \( u \) and \( v \) in \( Q_{n-1}^{0} \). Hence \( C \) is a hamiltonian cycle in \( Q_{n-1}^{0} \) passing through \( u \) and \( v \), and \( d_{C}(u,v) = h(u,v) = d \). Since \( n \geq 3 \), \( l(C) \geq 4 \). One may choose a adjacent vertex \( w \) in \( C \) such that the cycle \( C \) can be written as \( \langle u, S[u,v], v, P[v,w], w, u \rangle \) where \( l(S) = h(u,v) \) and \( l(P) \geq 1 \). Since \( (u, 0^d) \) and \( (w, 0^d) \) are two edges of dimension 0, \( 0^d \) and \( 0^d \) are two adjacent vertices in \( Q_{n-1}^{1} \). By Lemma 2, there exist paths formed by \( (u, 0^d), (w, 0^d), (w, 0^d), (u, 0^d) \) in the \( Q_{n-1}^{1} \) with length \( 1, 3, \ldots, 2^n \). Therefore, we can construct a cycle \( C' = (u, S[u,v], v, P[v,w], w, 0^d), (w, 0^d), (u, 0^d) \) containing the shortest path \( S[u,v] \). It is observed that \( C' \) is a geodesic cycle with even length of \( l(C') = 2^{n-1} + 1 + 2 + l(R) \) between \( u \) and \( v \) in \( Q_n \), where \( l(R) = 1, 3, \ldots, 2^n - 1 \). Therefore, \( 2^{n-1} + 2 \leq l(C') \leq 2^n \). The proof of this subcase is completed.

Subcase 2-2: \( u \in Q_{n-1}^{0} \) and \( v \in Q_{n-1}^{1} \) or \( v \in Q_{n-1}^{0} \) and \( u \in Q_{n-1}^{1} \). Without loss of generality, we may assume that \( u \in Q_{n-1}^{0} \) and \( v \in Q_{n-1}^{1} \). Applying Lemma 3, there exists a shortest path \( S[u,v] \) joining \( u \) and \( v \) in \( Q_n \) with all vertices (except \( u \)) in \( Q_{n-1}^{0} \). We write the path \( S[u,v] = \langle u, P[u, 0^d], v^0, v \rangle \). Hence \( l(P[u, 0^d]) = d - 1 \). Meanwhile, there exists a shortest path \( S_1[v,u] \) joining \( v \) and \( u \) in \( Q_n \) with all vertices (except \( u \)) in \( Q_{n-1}^{1} \). We write the path \( S_1[v,u] = \langle v, P_1[v, 0^d], u^0, u \rangle \). Then, \( l(P_1[v, 0^d]) = d - 1 \). Therefore, \( \langle u, 0^d, P[u, 0^d], v^0, v, P_1[v, 0^d], u^0, u \rangle \) forms a geodesic cycle of length \( 2d \) between \( u \) and \( v \) in \( Q_n \). If \( d = 2 \), the geodesic cycle of length 4 between \( u \) and \( v \) exists.

Since \( u \) and \( v \) are in \( Q_{n-1}^{0} \), by induction hypothesis, \( Q_{n-1}^{0} \) is geodesic bipancyclic. Let \( C \) be a geodesic cycle with even length of \( k \) satisfying \( \max\{2(d - 1), 4\} \leq k \leq 2^{n-1} \) between \( u \) and \( v \) in \( Q_{n-1}^{0} \). The cycle \( C \) can be rewritten as \( \langle u, S[u,v], v^0, w, R[w,u], u \rangle \) where \( l(S[u,v]) = d - 1 \) and \( l(R[w,u]) = k - d \). Hence \( \langle u, S[u,v], v^0, v \rangle \) is a shortest path of length \( Q_n \) joining \( u \) and \( v \). Since \( v^0 \) and \( w \) are adjacent, \( v^0 \) and \( v \) are adjacent in \( Q_{n-1}^{1} \). By Lemma 2, there are paths formed by \( \langle v, P[v, 0^d], 0^d, w, R[w,u], u \rangle \) in the \( Q_{n-1}^{1} \) with length \( m = 1, 3, \ldots, 2^n - 1 \). Therefore, we can construct a cycle \( C' = \langle u, S[u,v], v^0, v, P[v, 0^d], w^0, w, R[w,u], u \rangle \) containing the shortest path \( S[u,v] \). It is observed that \( C' \) is a geodesic cycle with even length of \( l(C') = d - 1 + 1 + m + 1 + k - d = m + k + 1 \) between \( u \).
and \( v \) in \( Q_n \). Therefore, \( \max\{2d, 6\} \leq l(C') \leq 2^n \).

The theorem is proved. \( \square \)

4 Balanced bipancyclic of Hypercubes

\( Q_n \) is balanced bipancyclic if for each pair of vertices \( u, v \in V(Q_n) \), it contains a balanced cycle of every even length of \( 2l \) satisfying \( \max\{h(u, v), 2\} \leq 2^{n-1} \) between \( u \) and \( v \). The following lemma is useful in the proof of Theorem 2.

Lemma 4 For any two disjoint edges \( (u, v) \) and \( (w, z) \) in \( Q_n \), with \( n \geq 2 \), there exist two disjoint paths, \( (u, P_1[u, v], v) \) and \( (w, P_2[w, z], z) \), in \( Q_n \) with equal length \( k \) where \( k = 1, 3, 5, 7, \ldots, 2^{n-1} - 1 \).

Proof. We prove this lemma by induction on \( n \).

Obviously, the lemma holds for \( n = 2 \). Assume that the lemma is true for every integer \( 2 \leq m < n \). Suppose that \( (u, v) \) is an edge of dimension \( i \) and \( (w, z) \) is an edge of dimension \( j \) where \( 0 \leq i, j \leq n - 1 \). Since \( n \geq 3 \), there exists an integer \( r \) such that \( r \neq i \) and \( r \neq j \). We may partition \( Q_n \) along dimension \( r \) into two \((n-1)\)-subcubes such that \( Q_{n-1}^0 \) denotes the subgraph of \( Q_n \) induced by \( \{x \in V(Q_n) \mid x_r = 0\} \) and \( Q_{n-1}^1 \) denotes the subgraph of \( Q_n \) induced by \( \{x \in V(Q_n) \mid x_1 = 1\} \). Since \( r \neq i \) and \( r \neq j \), \((u, v)\) and \((w, z)\) are in the same subcube or in different subcubes. The proof is divided into two cases: \((u, v)\) and \((w, z)\) lie on the same subcube \( Q_{n-1}^0 \) (or \( Q_{n-1}^1 \)), and \((u, v)\) and \((w, z)\) are in different subcubes.

Case 1: \((u, v)\) and \((w, z)\) lie on the same subcube \( Q_{n-1}^0 \) (or \( Q_{n-1}^1 \)).

Without loss of generality, we may assume that \((u, v)\) and \((w, z)\) lie on \( Q_{n-1}^0 \). By induction hypothesis, there exist two disjoint paths \( P_1[u, v] \) and \( P_2[w, z] \) in \( Q_{n-1}^0 \) with equal length \( k \) where \( k = 1, 3, 5, 7, \ldots, 2^{n-2} - 1 \). Let \( R_1[u, v] \) and \( R_2[w, z] \) be two disjoint paths of length \( 2^{n-2} - 1 \) in \( Q_{n-1}^0 \). Since \( n \geq 3 \), \( l(R_1) = l(R_2) \geq 1 \). We can rewrite \( R_1[u, v] \) (respectively, \( R_2[w, z] \)) as \((u, x, S_1[x, v], v)\) (respectively, \((w, y, S_2[y, z], z)\)) where \( l(S_1) = l(S_2) \geq 0 \), and \( x = v \) if \( l(S_1) = 0 \) (respectively, \( y = z \) if \( l(S_2) = 0 \)). Let \((u, u'), (x, x'), (w, w'), \) and \((y, y')\) be four disjoint edges of dimension \( r \). Hence \((u', x')\) and \((w', y')\) are two edges lying \( Q_{n-1}^0 \). By the induction hypothesis, there exist two disjoint paths \( T_1[u', x'] \) and \( T_2[w', y'] \) in \( Q_{n-1}^0 \) with equal length of \( m \) where \( m = 1, 3, 5, 7, \ldots, 2^{n-2} - 1 \). Therefore, two paths can be constructed as \( P_1 = \langle u, u', T_1[u', x'], x', x, S_1[x, v], v \rangle \) and \( P_2 = \langle w, w', T_2[w', y'], y', y, S_2[y, z], z \rangle \) where \( l(P_1) = l(P_2) = 2^{n-2} + m \) and \( m = 1, 3, 5, 7, \ldots, 2^{n-2} - 1 \). Hence \( P_1[u, v] \) and \( P_2[w, z] \) are two disjoint paths with equal length of \( 2^{n-2} + 1, 2^{n-2} + 3, 2^{n-2} + 5, \ldots, 2^{n-1} - 1 \).

Case 2: \((u, v)\) and \((w, z)\) are in different subcubes.

Applying Lemma 2, there are paths formed by \((u, P_1[u, v], v)\) in the \( Q_{n-1}^0 \) with length 1, 3, 5, 7, \ldots, \( 2^{n-1} - 1 \) and there are paths formed by \((w, P_2[w, z], z)\) in the \( Q_{n-1}^1 \) with length 1, 3, 5, 7, \ldots, \( 2^{n-1} - 1 \). The Lemma is proved. \( \square \)

Theorem 2 \( Q_n \) is balanced bipancyclic if \( n \geq 2 \).

Proof. Let \( u = u_{n-1}u_{n-2} \ldots u_1u_0 \) and \( v = v_{n-1}v_{n-2} \ldots v_1v_0 \) be any two distinct vertices of \( Q_n \) and \( h(u, v) = d \). To prove the theorem, we will find every balanced 2l-cycle between \( u \) and \( v \) where \( \max\{d, 2\} \leq \ell \leq 2^{n-1} - 1 \). The proof is divided into two parts: \( d = 1 \) and \( d \geq 2 \).

Case 1: \( d = 1 \), i.e. \( u \) and \( v \) are adjacent.

Without loss of generality, we may assume that \((u, v)\) is an edge of dimension 0. We may partition \( Q_n \) along dimension 1 into two \((n-1)\)-subcubes such that \( Q_{n-1}^0 \) denotes the subgraph of \( Q_n \) induced by \( \{x \in V(Q_n) \mid x_1 = 0\} \) and \( Q_{n-1}^1 \) denotes the subgraph of \( Q_n \) induced by \( \{x \in V(Q_n) \mid x_1 = 1\} \). Therefore, \( u \) and \( v \) are in the same subcube \( Q_{n-1}^0 \) or \( Q_{n-1}^1 \). Without loss of generality, we suppose that \( u \) and \( v \) are in \( Q_{n-1}^0 \).

Let \((u, u')\) and \((v, v')\) be two edges of dimension 1. Hence \( h(u, v') = 1 \) and \( u', v' \in V(Q_{n-1}^0) \). Applying Lemma 2, there are paths formed by \((u, P_1[u, v], v)\) in the \( Q_{n-1}^0 \) with length \( k_1 = 1, 3, 5, 7, \ldots, 2^{n-1} - 1 \) and there are paths formed by \((v', P_2[v', u'], u')\) in the \( Q_{n-1}^0 \) whose lengths are \( k_2 = 1, 3, 5, 7, \ldots, 2^{n-1} - 1 \). We can construct a cycle as \( C = \langle u, P_1[u, v], v, v', P_2[v', u'], u \rangle \) of length \( l(C) = k_1 + k_2 + 2 \) where \( k_1 = l(P_1) \) and \( k_2 = l(P_2) \). Obviously, the cycle \( C \) passes through \( u \) and \( v \).

(a) Balanced \((2k + 2)\)-cycle between \( u \) and \( v \) where \( k = 1, 3, 5, \ldots, 2^{n-2} - 1 \). Let \( k_1 = k \) and \( k_2 = k \). Then, \( l(C) = 2k + 2 \) where \( k = 1, 3, 5, \ldots, 2^{n-1} - 1 \). Hence \( d_C(u, v) = k = \frac{l(C)}{2} - 1 \). Since \( d \) is odd, \( \frac{l(C)}{2} \) is even, and \( d_C(u, v) = \frac{l(C)}{2} - 1 \), the cycle \( C \) is a balanced \((2k + 2)\)-cycle between \( u \) and \( v \) where \( k = 1, 3, 5, \ldots, 2^{n-1} - 1 \).

(b) Balanced \((2k + 4)\)-cycle between \( u \) and \( v \) where \( k = 1, 3, 5, \ldots, 2^{n-1} - 3 \). Let \( k_1 = k + 2 \) and \( k_2 = k \). Then, \( l(C) = 2k + 4 \) where \( k = 1, 3, 5, \ldots, 2^{n-1} - 3 \). Hence \( d_C(u, v) = k + 2 = \frac{l(C)}{2} \). Since \( d \)
is odd. \( \frac{C}{C} \) is odd, and \( d_C(u, v) = \frac{C}{C} \), the cycle \( C \)

is a balanced \((2k + 4)\)-cycle between \( u \) and \( v \) where

\( k = 1, 3, 5, \ldots, 2^{n-1} - 3 \).

**Case 2**: \( d \geq 2 \), i.e. \( u \) and \( v \) are not adjacent.

We prove this case by induction on \( n \). Obviously, the proof of case 2 holds for \( n = 2 \). Assume that the proof of case 2 is true for every integer \( 2 \leq m < n \). Let \( u = u_{n-1}u_{n-2} \ldots u_{n-2}u_{n-1} \) and \( v = v_0v_1 \ldots v_{n-1}v_0 \). Any two distinct vertices of \( Q_n \) and \( h(u, v) = d \). Partitioning \( Q_n \) along dimension \( 0 \), \( Q_n \) can be divided into two \((n - 1)\)-subcubes where \( Q_{n-1} \) denotes the subgraph of \( Q_n \) induced by \( \{ x \in V(Q_n) \mid x_0 = 0 \} \) and \( Q_{n-1} \) denotes the subgraph of \( Q_n \) induced by \( \{ x \in V(Q_n) \mid x_0 = 1 \} \).

**Subcase 2-1**: \( u, v \in Q_{n-1} \) or \( u, v \in Q_{n-1} \).

Without loss of generality, we may assume that \( u, v \in Q_{n-1} \). For the basis of this proof, we consider \( Q_3 \). It is clear that \( Q_3 \) is balanced bipancyclic.

**Suppose that** \( n \geq 4 \). By induction hypothesis, \( Q_{n-1} \) is balanced bipancyclic. Every balanced \( 2l \)-cycle between \( u \) and \( v \) in \( Q_n \) can be found in \( Q_{n-1} \) such that \( d \leq l \leq 2^{n-2} \). Let \( C \) be a balanced \( m \)-cycle with \( m \geq 6 \) between \( u \) and \( v \) in \( Q_{n-1} \). Hence we rewrite the cycle \( C \) as \( (u, x, P_1[x, v], y, P_2[y, u], \ldots, u) \). Let \( (u, v_0), (x, x_0), (v, v_0), (y, y_0) \) be four edges of dimension 0. It is observed that \( (u, v_0), (x, x_0) \), and \( (v, y_0) \) are four distinct vertices in \( Q_{n-1} \), and that \( (u, x), (x, v_0) \), and \( (y, v) \) are two disjoint edges in \( Q_{n-1} \). Applying Lemma 4, there exist two disjoint paths \( S_1[u_0, x_0, y_0] \) and \( S_2[v_0, x_0, y_0] \) in \( Q_{n-1} \) such that \( l(S_1) = l(S_2) = k \) where \( k = 1, 3, 5, 7, \ldots, 2^{n-2} - 1 \). Therefore, we may construct a cycle \( C' \) as \( (u, v_0), S_1[u_0, x_0, y_0], x_0, x, P_1[x, v], v, x_0, S_2[v_0, y_0], y_0, y, P_2[y, u], \ldots, u) \) passing through \( u \) and \( v \). Hence \( l(C') = m + 2k + 2 \).

**Subcase 2-1-1**: Balanced \((2^{n-1} + 2k)\)-cycle between \( u \) and \( v \) where \( k = 1, 3, 5, \ldots, 2^{n-2} - 1 \). Let \( m = 2^{n-1} - 2 \). Therefore, \( l(C') = 2^{n-1} + 2k \).

(a) Suppose that \( d \) is odd. Since \( C \) is a balanced \((2^{n-1} - 2)\)-cycle between \( u \) and \( v \), and \( \frac{C}{C} = 2^{n-2} - 1 \) is odd, \( d_C(u, v) = 2^{n-2} - 1 \). It is clearly that \( d_{C'}(u, v) = d_{C}(u, v) + k + 1 = 2^{n-2} + k + 1 \) and \( l(C') = 2^{n-2} + 2k + 1 \). Since \( d \) is odd, \( \frac{C}{C} = 2^{n-2} + 2k + 1 \) is odd, and \( d_{C'}(u, v) = 2^{n-2} + k + 1 = \frac{C}{C} - 1 \). Therefore, \( l(C') = 2^{n-1} + 2k + 2 \).

(b) Suppose that \( d \) is even. Since \( C \) is a balanced \((2^{n-1} - 2)\)-cycle between \( u \) and \( v \), and \( \frac{C}{C} = 2^{n-2} - 1 \) is odd, \( d_C(u, v) = 2^{n-2} - 2 \). It is clearly that \( d_{C'}(u, v) = d_{C}(u, v) + k + 1 = 2^{n-2} + k + 1 \) and \( l(C') = 2^{n-2} + k + 1 \). Since \( d \) is even, \( \frac{C}{C} = 2^{n-2} + k + 1 \) is odd, and \( d_{C'}(u, v) = 2^{n-2} + k + 1 = \frac{C}{C} - 1 \). Therefore, \( l(C') = 2^{n-1} + 2k + 1 \).

**Subcase 2-2-1**: Balanced \((2^{n-1} + 2k)\)-cycle between \( u \) and \( v \) where \( k = d - 1, d + 1, d + 3, \ldots, 2^{n-1} - 1 \). Let \( k_1 = k_2 = k \) where \( k = d - 1, d + 1, d + 3, \ldots, 2^{n-1} - 1 \). Therefore, \( l(C') = 2k + 2 \). One can observe that \( \frac{C}{C} = k + 1 \) and \( d_{C}(u, v) = k + 1 \). Since \( d \) is even, \( \frac{C}{C} = 2k \) is even, and \( d_{C'}(u, v) = \frac{C}{C} \). Therefore, \( C' \) is a balanced \((2^{n-1} + 2k - 2)\)-cycle between \( u \) and \( v \) where \( k = d - 1, d + 1, d + 3, \ldots, 2^{n-1} - 3 \). Therefore, \( l(C) = 2k + 4 \). One can observe that \( \frac{C}{C} = k + 2 \) and \( d_{C}(u, v) = k + 1 \).
Since $d$ is even, $l(C)$ is odd, and $d_C(u, v) = \frac{l(C)}{2} - 1$, the cycle $C$ is a balanced $(2k + 4)$-cycle between $u$ and $v$ where $k = d - 1, d + 1, d + 3, \ldots, 2^{n-1} - 3$.

Subcase 2-2-2: $d$ is odd, i.e., $u$ and $v$ are in different partite sets. Hence $v^0$ and $u$ are in the same partite set. Similarly, $v^0$ and $u$ are in the same partite set. By Lemma 2, there exists a path $P_1[u, v^0]$ (respectively, $P_2[v, u^0]$) connecting $u$ and $v^0$ (respectively, $v$ and $u^0$) where $l(P_1) = d - 1, d + 1, d + 3, 2^{n-1} - 2$ (respectively, $l(P_2) = d - 1, d + 1, d + 3, \ldots, 2^{n-1} - 2$). The cycle $C$ can be constructed as $(u, P_1[u, v^0], v^0, v, P_2[v, u^0], u^0, u)$. Therefore, the cycle $C$ passing through $u$ and $v$, and $l(C) = k_1 + k_2 + 2$ where $k_1 = l(P_1)$ and $k_2 = l(P_2)$.

(a). Balanced $(2k + 2)$-cycle between $u$ and $v$ where $k = d - 1, d + 1, d + 3, \ldots, 2^{n-1} - 2$. Let $k_1 = k$ and $k_2 = k$ where $k = d - 1, d + 1, d + 3, \ldots, 2^{n-1} - 2$. Therefore, $l(C) = 2k + 2$. One can observe that $\frac{l(C)}{2} = k + 1$ and $d_C(u, v) = k + 1$. Since $d$ is odd, $\frac{l(C)}{2}$ is odd, and $d_C(u, v) = \frac{l(C)}{2} - 1$, the cycle $C$ is a balanced $(2k + 2)$-cycle between $u$ and $v$ where $k = d - 1, d + 1, d + 3, \ldots, 2^{n-1} - 2$.

(b). Balanced $(2k + 4)$-cycle between $u$ and $v$ where $k = d - 1, d + 1, d + 3, \ldots, 2^{n-1} - 4$. Let $k_1 = k + 2$ and $k_2 = k$ where $k = d - 1, d + 1, d + 3, \ldots, 2^{n-1} - 4$. Therefore, $l(C) = 2k + 4$. One can observe that $\frac{l(C)}{2} = k + 2$ and $d_C(u, v) = k + 1$. Since $d$ is odd, $\frac{l(C)}{2}$ is even, and $d_C(u, v) = \frac{l(C)}{2} - 1$, the cycle $C$ is a balanced $(2k + 4)$-cycle between $u$ and $v$ where $k = d - 1, d + 1, d + 3, \ldots, 2^{n-1} - 4$.

(c). Balanced $2^n$-cycle between $u$ and $v$. Let $w \in V(Q_{n-1}^1)$ and $h(w, v) = 1$. It is observed that $h(w, v^0)$ is odd. By Lemma 1, there exists a path $P[w, v^0]$ of length $2^{n-1} - 2$ joining $v$ and $w^0$ passing all vertices of $Q_{n-1}^1$ except $w$. Let $(w, w^0)$ be an edge of dimension 0. Then, $P[w, v^0]$ in $Q_{n-1}^0$, and $w^0$ and $u$ are in different partite sets. By Lemma 2, there exists a hamiltonian path $P_1[u, w^0]$ joining $u$ and $w^0$ in $Q_{n-1}^0$. Therefore, the longest cycle $C$ between $u$ and $v$ in $Q_n$ can be constructed as $(u, P_1[u, w^0], w^0, v, P_2[v, u^0], u^0, u)$. Therefore, the cycle $C$ passing through $u$ and $v$, such that $l(C) = 2^{n-1} - 1 + 1 + 2^{n-1} - 2 + 1 = 2^n$ and $d_C(u, v) = 2^{n-1} - 1 = \frac{l(C)}{2} - 1$. Since $d$ is odd, $\frac{l(C)}{2}$ is even, and $d_C(u, v) = \frac{l(C)}{2} - 1$, the cycle $C$ is a balanced cycle between $u$ and $v$. The theorem is proved.

5 Conclusions

For any two vertices $u$ and $v$, the transmission delay from $u$ to $v$ is minimum in a geodesic cycle. We showed that for any two vertices $u$ and $v$ in $Q_n$, there exists a geodesic cycle of every even length of $k$ satisfying $max\{2h(u, v), 4\} \leq k \leq 2^n$. To route a packet from $u$ to $v$ in a cycle, one may first breaks the packet into two smaller pieces. Then, route the two pieces along two internal vertex-disjoint paths to destination $v$. The packet is combined in $v$ until these two pieces arrived. It is of interest to find a cycle passing through $u$ and $v$ such that lengths of two disjoint paths between $u$ and $v$ in this cycle are as equal as possible. We prove that for any two vertices $u$ and $v$, there exists a balanced cycle of every even length of $k$ satisfying $max\{2h(u, v), 4\} \leq k \leq 2^n$ in $Q_n$.

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References:


